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SOLUTION TO MATHEMATICAL MODEL ON MALARIA TRANSMISSION DYNAMICS USING HOMOTOPY PERTURBATION METHOD (HPM)

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ABSTRACT
Mathematical models have been used to provide an explicit framework for understanding malaria transmission dynamics in human population for over 100 years, with the disease still thriving and threatening to be a major source of death and disability due to changed environmental and socio- economic conditions. In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical, and engineering problems both linear and non-linear. In this work, we used the Homotopy Perturbation Method (HPM) to obtain the analytic solution of the differential equations of the (SIR-SI) mathematical model and we apply the Bellman and Cooke's theorem of stability to verify the stability of the model at equilibrium state. This work confirms the power, simplicity and efficacy of the method, also this method is a suitable method for solving any partial differential equations or system of partial differential equations as well.

Definition of Variables

S = Susceptible human	I = Infected human	R = Recovered human/Removed human
I_v = Infected vector	π= human birth rate	λ= vector birth rate
β= Contact rate	γ =Natural death rate	α = Death rate due to disease

δ= Transmission rate between susceptible vector and infected human
ω = Transmission rate between susceptible vector and infected human

INTRODUCTION
Malaria is a disease that causes fever and shivering (shaking of the body) caused by the bite of some types of mosquito called Anopheles. (Oxford Advanced Learners Dictionary) Malaria is an ancient disease having a huge social, economic and health burden. It is predominantly present in tropical countries. It is a disease mainly found in tropical areas as sub-Saharan Africa, Central and South America, South East, Asia and the Pacific Islands which are called malaria regions. Even though the diseases have been investigated for hundreds of years, it

remains a major public health problem with 109 countries declared as endemic to the disease (Sandip Mandal et al 2008). It is a life threatening disease caused by parasites called plasmodium that are transmitted to people through the bite of infected anopheles mosquito (WHO, 2009). The female anopheles mosquito gets infected when it bites someone carrying the malaria parasite. There are four different types of plasmodium parasite: plasmodium falciparium is the only parasite which causes malignant malaria. It causes symptoms straight away and can be mild or severe. Secondly, plasmodium vivax causes benign (non cancerous) malaria with less severe symptoms. The vector can stay in the liver for up to three years and can lead to relapse. Thirdly, plasmodium malariae also causes benign malaria and is relatively rare. Lastly plasmodium ovale also causes benign malaria and can stay in the blood and liver without causing symptoms. Plasmodium falciparium is responsible for about three-quarters of reported malaria cases. Most of the other cases of malaria are caused by plasmodium vivax with a few caused by the other species (Lallo et al, 2007). It is possible to get infected with one type of plasmodium parasite. Each parasite causes a slightly different type of illness.

Malaria is a vector-borne infectious disease which is caused by protozoan parasite. **Note:** Malaria shares many characteristics with other protozoan parasites, which cause disease such as African trypanosomiasis and visceral leishmaniaasis. Malarial infection is cyclic, so I can start the circle anywhere. In this case an infected female mosquito bites a human host and infects the human with some of her saliva. The saliva acts as a pain killer so that the human will not feel the bite and because the female mosquito is infected; her saliva contains the sporozite form of the malaria parasites. All blood in a human's body is filtered by the liver, and here the sporozite will reproduce in a sexual manner and forming large quantities of the trophozoite form of the parasite. These trophozoites are released into the blood stream where the red blood cells are invaded. The parasite uses the red blood cell as an incubator and again reproduces, producing the merozite form of the parasite in quantities large enough to cause the cell to rupture. Malaria symptoms are characterized by high fever, chills, flu-like symptoms and in many cases, death. The common first symptoms of malaria are similar to the flu. The patient may have: a headache, aching muscles, tummy ache and weakness or lack of energy. A day or three about later, the body temperature may rise (up to 10 degree Celsius) and the patient may have: a fever, shivers, mild chills, severe headache, vomiting, diarrhea and loss of appetite (Bupa, 2009), however, it takes at least 6 days for symptoms to appear. The time it takes for symptoms to appear can vary with type of parasite that mosquito was carrying. If the person gets infected with plasmodium falciparium, malaria can progress to more severe form called complicated malaria. The following symptoms may appear: low blood sugar levels, severe anaemia, jaundice, fluid on one's lung (Pulmonary Oedema), acute respiratory distress syndrome, kidney failure, spontaneous bleeding, and state of shock (circulatory collapse), fits (convulsion), paralysis and coma. Severe malaria can affect the patient's brain and central nervous system and can be fatal (Bupa, WHO, 2009).

Natural immunity partially protects a limited number of people. Prevention is the only key to wiping out malaria, keeping mosquitoes away from people, either by physically blocking or by eliminating them from the environment. There are many ways to do this. Countries affected by malaria use a multifaceted approach to stopping this scourge. First, anti-malaria drugs can be given to prevent this epidemic, particularly in areas with small outbreaks. The second approach uses mosquito control to prevent outbreaks. Third approach, still under development, will prevent the disease through vaccination.

Malaria, especially *falciparium* malaria, is a medical emergency that requires a hospital stay. *Chloroquine* is often used as an anti-malarial medication. However, *chloroquine*-resistant infections are common in some parts of the world. Possible treatments for *chloroquine*-resistant infection include: (i) The combination of *quinidine* or quinine plus *doxycycline*, tetracycline, or *clindamycin*. (ii) *Atovaquone* plus *proquanil* (*malarone*) (iii) *Mefloquine* or *artesunate* (iv) The combination of *pyrimetnhanine* or *sulfadoxine* (*fansidar*). The choice of medication depends in part on where and when you were infected. Medical care, including fluids through a vein (iv) and other medications and breathing (respiratory) support may be needed. The aim of this project research is to construct a mathematical model on malaria diseases dynamics by use of the system of first order differential equations with five subpopulations of Susceptible, S(t), infected, I(t), and Recovered, R(t),susceptible, S(t), infected, I(t). Using the He's Homotopy perturbation method for finding solution to the modeled equations. The objective is to solve the mathematical model equations analytically using Homotopy Perturbation Method and obtain the conditions for the stability of the disease spread and the disease free equilibrium (DS&DFE) states respectively. The health as well as the socio-economic impacts of emerging and re-emerging vector born disease like mosquito is significant. The disease is endemic and claims so many lives and consequently makes its study valuable. Early Treatment of malaria shortens its duration, prevent complications and avoid majority of deaths.

Beyond the human toll, malaria wrecks significant economic havoc in high-rate areas decreasing Gross Domestic Product (GDP) by as much as 1.3% in countries with high levels of transmission. Over the long term, these aggregated annual losses have resulted in substantial differences in GDP between countries both personal and public expenditures on prevention and treatment. In some heavy-burden countries, the disease accounts for, up to 40% public health expenditures, 30% to 50% of impatient hospital admissions up to 60% of outpatient health clinic visits (WHO, 2009). Malaria disproportionately affects poor people who cannot afford treatment or have limited access to health care, traps families and communities in a down spiral of poverty. As a result of this a mathematical study has been done as a way of guidance for decision makers on which intervention to focus on. In this work, the subpopulations of the human host and the mosquito host is divided into susceptible humans (S_h), infected humans (I_h), recovered humans, (R_h) susceptible vector (S_v) and infected vector (I_v). Birth is not constant in the susceptible, infected and recovered can be removed either through natural death or death by the disease respectively. The individuals move from susceptible class to infected class by interaction with the infected hosts, and the infected moves to the removed class by treating the disease. The (SIR-SI) flow chart of malaria helps us to clearly see the movement of each population.

GENERAL OVERVIEW OF MALARIA MODELS

Carter (2002) "spatial simulation of malaria transmission and its control transmission blocking vaccination". Given the challenge of malaria be it in endemic area of otherwise, predictive mathematical modeling and computer simulation remains our greatest hope. Ngwa et al (2000) "A mathematical model for endemic with variable and mosquito populations", formulated a variable humans and mosquitoes mathematical model consisting of Susceptible- Exposed- infectious- Recovered- Susceptible (SEIRS) pattern for humans, and susceptible- Exposed - infectious (SEI) pattern for mosquitoes. The primary objective was to study endemic malaria and the consequent diseases related deaths in endemics regions. The importance of including demographic effects with net population growth was seen to enable the model to predict the number of facilities that may arise as a result of malaria. This type of prediction is not evident in the constant population model and hence has been over looked in previous mode for malaria. C.J Silva (2013)," an optimal control approach to malaria prevention via insecticide- treated nets" formulated a recent mathematical model for the effect of insecticide Treated Nets (ITNS) on the transmission dynamics of malaria infection, which takes into account the humans behavior. He introduced a model supervision control, representing International Electro-technical commission (IEC) campaign for improving the ITN usage. He proposed and solves an optimal control problem where the aim is to minimize the number of infectious humans while keeping the low cost.

Also (S. Mandal, 2011) developed a mathematical model for malaria that incorporate global warming and local socio-economic conditions. The main objective was to apply sensitive analysis to a mathematical model describing malaria transmission relating global warming and local socio-economic conditions which represent the level of malaria infection in a community. Their work was mainly based on the infection and none of the interventions were tackled. Gomez-Elipe et al, (2003), "Forecasting incidence base on monthly case report and environmental factors" studied a mathematical model involving malaria incidence in an area of unstable transmission by studying the association between environmental variables such as rainfall, temperature and vegetation density, and disease dynamics, malaria control were not mentioned. Li J. (2008) "A malaria model with immunity in humans", who formulated a mathematical model for malaria transmissions that includes, incubation periods for both infected human hosts and mosquitoes. It was demonstrated that models having the same reproductive number but different number of progression stages can exhibit different transient transmission dynamics. It was concluded that humans acquire partial immunity to malaria after infection although the mechanisms of immunity are not fully understood. The acquired immunity appears to depend on both the duration and the intensity of past exposure to infection.

MB Abdullahi (2013), presented an ordinary differential equation mathematical model for the spread of malaria in human and mosquito population. They assumed that both species follow logistic population model, with immigration and disease induced death of humans. The sophisticated of the epidemiological modeling efforts has grown steadily. Smith and Hove-Musekwa (2008), "Determining effective spraying period to control malaria via indoor spraying in sub-Sahara Africa " developed a mathematical model for both regular and non-fixed spraying using impulsive differential equations in order to determine the minimal effective spraying period, as well as the amount by which mosquitoes should be reduced at each spraying events. The effects of climate change on the prevalence of mosquitoes were considered. The result showed that both regular and non-fixed spraying result in a significant reduction in the overall number of mosquitoes, as well as the number of malaria cases in humans. However, only one intervention was discussed. Comparative knowledge of the efficacy of different control strategies is necessary to design useful and cost effective malaria control programs. Mathematical modeling of malaria can play a unique role in comparing effects of control strategies. The researcher, therefore, investigate the effects of such control measures on malaria dynamics and their costs.

MODEL DESCRIPTION AND METHODOLOGY

Modeling of Malaria Incidence

A basic model for the epidemiology of malaria outbreaks is primarily based on portioning a given subpopulation of human host into classes, though the allocation of individuals to any of the classes is not static, that is any of the host is free to move from one class to another at any given time. However, whenever we make mention of one being susceptible, it implies that exist an illness or infection which one is likely to contract class. It is actually this illness or infection that moves a susceptible host to infected class, and vice versa. Now consider the epidemic model of an intrinsic population of host N, subjected to certain infection (mosquito borne); here the host is well mixed, meaning that it is equally probable for any two hosts to come into contact. Suppose there is availability of female Anopheles mosquitoes and we wish to build our assumptions around the intrinsic population, then the following assumptions holds.

i) Population size $\frac{ds}{dt} + \frac{dI}{dt} + \frac{dR}{dt} = 0 \Rightarrow s(t) + I(t) + R(t) = N$
   Is built into the system (There is addition to the system by either migration or by birth).
ii) The disease is fatal (that is it lead to death), and hence natural death due to illness depletes the number of susceptible and infected host respectively.
iii) The susceptible host has no form of immunity covering them. But infected class can be vaccinated.
iv) The infective period of the vector ends with its death.
v) The incubation period of the infection is very short so that transmission is very short so that transmission is not delayed.

The Basic SIR-SI Model

By convention, the total human population N at any time t of a standard (SIR-SI) malaria models is basically divided into three subpopulation of susceptible human at any time t, denoted by S(t), individuals infected with malaria as I(t) and recovered individuals from malaria as R(t). The letters S(t), I(t), and R(t) represent the number of people in each class at a particular time. We make the letters (S), (I) and

(R) a function of t to indicate variation over time. The implication of the variables (S), (I), and (R) being a function of t is because of the dynamic nature of the models, in that the numbers in each compartment may fluctuate over time. "The importance of this dynamic aspect is most obvious in an endemic disease with a short infection period such as malaria. Such diseases tend to occur in cycles of outbreaks due to the variation in number of susceptible (S(t)) over time". For Example during an epidemic, the number of susceptible individuals falls rapidly as more of them are infected and thus enter the infectious, and through medication enters the recovered class. The progression of each number of the population from susceptible to infections can be seen clearly in the diagram below.

Model Formulation

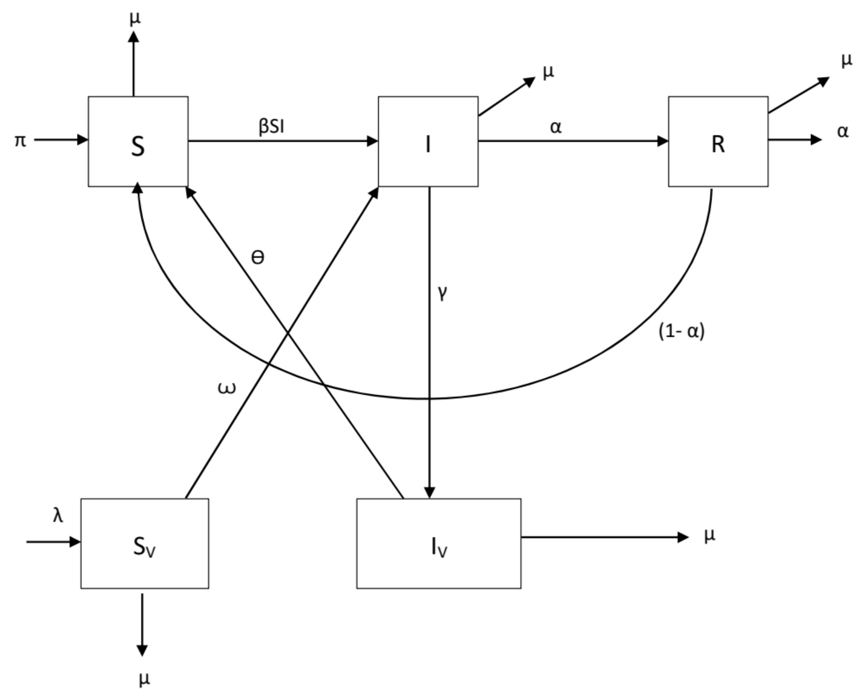

Figure 3.3: Schematic diagram for the flow of the (SIR-SI) model

The arrow sign signifies a transition and a mode of transmission of host and vectors from one compartment to another respectively. Between susceptible human, and the infected, the transition rate is βI, where β is the rate of contact; which is also a sure probability of getting the disease in a contact (mosquito bite) between a susceptible and an infectious individual.

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Mathematically we obtain the following differential equations:

i) $\frac{ds}{dt} = \pi - \beta S\, Iv - (I - \alpha)R - \theta Iv - \mu S$

ii) $\frac{dI}{dt} = \beta SI_v - \omega S_v - aI_v - \alpha R - \mu I$

iii) $\frac{dR}{dt} = \alpha R - \mu R - \alpha\mu - (1 - \alpha)R$

iv) $\frac{dS_v}{dt} = \lambda - \omega S_v - \mu S_v$

v) $\frac{dI_v}{dt} = \gamma I_v - \theta S - \mu I_v$

Where the parameters (I-α) = Recovery rate due to treatment

Solution of the Model

Analytic Solution Using Homotopy Perturbation Method (HPM)

Non-linear phenomena are very important in various fields of sciences and engineering. Most Non-linear models of real life problem are still very difficult to solve. Therefore, approximate analytic solutions such as (HPM) have been introduced. This method is the most effective and convenient one for both weakly and strong non-linear equations. Perturbation method mentioned is based on assuming a small parameter. The majority of non-linear problems, especially those that have strong non-linearity, have no small parameters at all and the approximate solutions obtained by the perturbation methods, in some cases, are valid only for the small parameter. Generally, the perturbation solutions are uniformly valid as long as a scientific system parameter is small. However we cannot rely fully on the approximations, because there is no criteria on which the small parameter should exist. Thus, it is essential to check the validity of the approximations numerically and/or experimentally. To overcome these difficulties (HPM) have been proposed recently.

Basic Idea of He's Homotopy Perturbation Method (HPM)

To illustrate the basic ideas of HPM, we consider the following non-linear differential equations.

$A(u) - f(r) = 0, \; r \in \Omega$ - - - (4.1)

With the boundary condition of;

$\beta(u, \frac{\partial u}{\partial n}) = r \in \Gamma$ - - - (4.2)

Where: A, B, f(r) are Γ are a different operator, a boundary operator, a known analytical function and boundary of the domain Ω respectively.

Generally speaking the operator A can be divided into a linear part L and non-linear part N. equation (4.1) can now be written as:

$L(u) - N(u) - f(r) = 0$ - - - (4.3)

By the Homotopy technique, we construct a Homotopy $v(r,P): \Omega \times [0,1], \rightarrow R$, which satisfies

$H(v,p) = (1 - p)[L(v) - L(u_0)] - P[A(v) - f(r)] = 0$ - - (4.4a)

$P \in [0,1], \Gamma \in \Omega$

Or

$H(v,p) = L(v) - L(u_0) - PL(U_0) - P[N(v) - f(r)] = 0$ - - (4.4b)

Where $P \in [0,1]$ is an embedding parameter, while u_0 is an initial approximation of equation (4.1), which satisfies the boundary conditions. Obviously, from equations (4.4a) and (4.4b) we will have

$H(v,0) = L(v) - L(u_0) = 0$ - - - - -(4.5)

$H(v,1) = A(v) - f(r) = 0$ - - - - -(4.6)

The changing process of p from zero to unity is just that of v(r,p) from $u_0(r)$. In topology this is called deformation, while $L(v) - L(u_0)$ and $A(v) - f(r)$ are called Homotopy.

According to the HPM, we can first use the embedding parameter P as a "small parameter", and assume that the solution of equations (4.4a) and (4.4b) can be written as a power series in P:

$V = V_0 - PV_1 - P^2V_2$ - - - - - (4.7)

Setting P = 1, yielding in the approximate solution of equation (4.1) to:

$U = \lim_{p=1} V = V - V_1 - V_2$ - - - - - (4.8)

The combination of the perturbation method and the Homotopy methods is called the HPM, which eliminates the drawbacks of the traditional perturbation methods. While keeping all its advantage, the series (4.8) is convergent for most cases.

However, the convergent rate depends on the non-linear operator A(v), moreover, he made the following suggestions:

1. The second derivations of N(v) with respect to V must be small because the parameter may be relatively large, i.e. $P \rightarrow 1$.

2. The norm of L^{-1} $\frac{\partial N}{\partial v}$ must be smaller than 1, so that the series converges.

Implementation of the Homotopy Perturbation Method

$$\frac{dS}{dt} = \pi - \beta\ S\ I_T - (I-\alpha)R - \theta I_T - \mu S \qquad - \quad - \quad - \qquad (4.1)$$

$$\frac{dI}{dt} = \beta\ S\ I_T \quad \omega S_T \quad \alpha I_T \quad \alpha R \quad \mu I \qquad - \quad - \quad - \qquad (4.2)$$

$$\frac{dR}{dt} = \alpha R - \mu R - \alpha\mu - (1-\alpha)R \qquad - \quad - \quad -- \qquad (4.3)$$

$$\frac{dS_T}{dt} = \lambda - \omega\ S_T - \mu S_T \qquad - \quad - \quad - \quad - \quad - \qquad (4.4)$$

$$\frac{dI_T}{dt} = \gamma I_T - \theta S - \mu I_T \qquad - \quad - \quad - \quad - \quad - \qquad (4.5)$$

With $S(0) = 0,\ I(0) = 0,\ R(0) = 0,\ S_T = 0,\ I_T(0) = 0,$ as initial conditions

Now equation $(4.1) \Rightarrow \frac{dS}{dt}\ \pi - \beta\ S\ I_T - (I-\alpha)R - \theta I_T - \mu S = 0.$

Where $\frac{dS}{dt} = 0$ (Linear part of (4.1)) or

$\Pi - \beta\ S\ I_T - (1-\alpha)R - \theta I_T - \mu S = 0$

Now equation $(4.2) \Rightarrow \frac{dI}{dt} = \beta\ S\ I_T - \omega S_T - \alpha I_T - \gamma R - \mu I = 0$

Where $\frac{dI}{dt} = 0$ (Linear part of (4.2)) or

$\beta\ S\ I_T - \omega S_T - \gamma I_T - \alpha R - \mu I = 0.$ (Non − linear part of (4.2))

Now equation $(4.3) \Rightarrow \frac{dR}{dt} = \alpha R - \mu R - \alpha\mu - (1-\alpha)R = 0.$

Where $\frac{dR}{dt} = 0$ (Linear part of (4.3)) or

$\alpha R - \mu R - \alpha\mu - (1-\alpha)R = 0.$ (Non − linear part of (4.3))

For equation $(4.4) \Rightarrow \frac{dS_T}{dt} = \lambda - \omega\ S_T - \mu S_T = 0.$

Where $\frac{dS_T}{dt} = \lambda - \omega\ S_T - \mu S_T = 0$, where $\frac{dS_T}{dt} = 0$ (Non − linear part of (4.4))

For $\frac{dI_T}{dt} = \gamma I_T - \theta S - \mu I_T = 0$

$\Rightarrow \frac{dI_T}{dt} = 0$ (Linear part of (4.5)) or $\gamma I_T - \theta S - \mu I_T = 0$ (Non − linear part of (4.5))

Now applying (HPM) to (4.1) we have

$$(1-P)\frac{dS}{dt} - P\left[\frac{dS}{dt} - \pi - \beta S I_T - (I-\alpha) - P\theta I_T - \mu S\right] = 0$$

Removing brackets we have

$$\Rightarrow \frac{dS}{dt} - P\pi - P\beta S I_T - P(1-\alpha)R - P\theta I_T - P\mu S = 0 \qquad - \quad - \quad - \qquad (4.6)$$

Let $S = x_0 - Px_1 - P^2x_2 - \ldots\ldots$ - - - - - (4.7)

$I = y_0 - Py_1 - P^2y_2 - \ldots\ldots$ - - - - - (4.8)

$R = z_0 - Pz_1 - P^2z_2 - \ldots\ldots$ - - - - - (4.9)

$S_v = m_0 - Pm_1 - P^2m_2 - \ldots\ldots$ - - - - - (4.10)

$I_v = n_0 - Pn_1 - P^2n_2 - \ldots\ldots$ - - - - (4.11)

Substituting (4.7) to (4.11) into equation (4.6) and expanding, we get

$\dot{x}_0 P\dot{x}_1 - P^2\dot{x}_2 - \ldots.$
$-P\pi - P\beta[(x_0Px_1 - P^2x_2 - \ldots.)(n_0Pn_1 - P^2n_2 - \ldots)] - P(1-\alpha)[z_0Pz_1 - P^2z_2 - \ldots.] -$
$P\theta(n_0 - Pn_1 - P^2n_2 - \ldots.) - P(x_0Px_1 - P^2x_2 - \ldots.) = 0$

Applying (HPM) to (4.2) we have

$(1-P)\frac{dI}{dt} - P[\frac{dI}{dt} - \beta S I_v - \omega S_v - \gamma I_v - \alpha R - \mu I] = 0.$

Removing brackets we have;

$\Rightarrow \frac{dI}{dt} - \beta S I_v - \omega S_v - \gamma I_v - \alpha R - \mu I] = 0.$ - - - - 4.15)

Substituting (4.7) to (4.11) into equation (4.15), we get,

$\dot{y}_0 - P\dot{y}_1 - P^2\dot{y}_2 - \ldots.$
$-P\beta(x_0 - Px_1 - P^2x_2 - \ldots.)(y_0 - Py_1 - P^2y_2 - \ldots.) - P\gamma(n_0 - Pn_1 - P^2n_2 - \ldots.) -$
$P\alpha(z_0 - Pz_1 - P^2z_2 - \ldots.) - P(y_0 - Py_1 - P^2y_2 - \ldots.) = 0$

Now, collecting the coefficient of the resultant equation based on powers of P-terms, we get;

$P^0 : \dot{y}_0 = 0$ - - - - - - - - (4.16)

$P^1 : \dot{y}_1 - \beta x_0 y_0 - \gamma n_0 - \alpha z_0 - \mu y_0 = 0$ - - - - - (4.17)

$P^2 : \dot{y}_2 - \beta(x_0y_1 - x_1y_0) - \gamma y_1 - \alpha z_1 - \mu y_1 = 0$ - - - - (4.18)

Rearranging equation (4.18)

$\Rightarrow P^2 : \dot{y}_2 - \beta(x_0y_1 - x_1y_0) - (\gamma - \mu)y_1 - \alpha z_1 = 0$ - - - (4.18)

Applying Homotopy perturbation method (HPM) to (4.3),

$(1-P)\frac{dR}{dt} - P\left[\frac{dR}{dt} - \alpha R - \mu R - \alpha\mu - (I-\alpha)R\right] = 0.$

Removing bracket, we have

$\frac{dR}{dt} - \alpha R - \mu(R - \alpha\mu - (1-\alpha)R = 0$ - - - -(4.19)

Substituting (4.7) to (4.11) into equation (4.19), we expand to get;

$\dot{z}_0 - P\dot{z}_1 - P^2\dot{z}_2 - \ \ldots\ - P\alpha(\dot{z}_0 - P\dot{z}_1 - P^2\dot{z}_2 - \ \ldots) - P\mu(\dot{z}_0 - P\dot{z}_1 - P^2\dot{z}_2 - \ \ldots) - P\alpha\mu - P(1-\alpha)(\dot{z}_0 - P\dot{z}_1 - P^2\dot{z}_2 - \ \ldots) = 0$

Now, collecting the coefficient of the resultant equations base on powers of P-terms, we get;

$P^0 : \dot{z}_0 = 0$ - - - - - - - - (4.20)

$P^1 : \dot{z}_1 - \alpha\dot{z}_0 - \mu\dot{z}_1 - (1-\alpha)\dot{z}_0 = 0 \Rightarrow P^1 : \dot{z}_1 - (\alpha - \mu - (1-\alpha))\dot{z}_0 = 0$ - -(4.21)

$P^2 : \dot{z}_2 - \alpha\dot{z}_2 - \mu\dot{z}_1 - (1-\alpha)\dot{z}_1 = 0 \Rightarrow P^2 : \dot{z}_2 - (\alpha - \mu - (1-\alpha))\dot{z}_1 = 0$ - - (4.22)

Applying Homotopy perturbation method (HPM) to (4.4),

$(1-P)\frac{dS_r}{dt} - P\left[\frac{dS_r}{dt} - \lambda - \omega S_r - \mu S_r\right] = 0$

Removing bracket we get

$\frac{dS_r}{dt} - P\lambda - P\omega - S_r - PS_r = 0$ - - - -- - (4.23)

Substituting (4.7) to (4.11) into equation (4.23), we expand to get;

$\dot{m}_0 - P\dot{m}_1 - P^2\dot{m}_2 - \ \ldots\ - P\lambda - P\omega(\dot{m}_0 - P\dot{m}_1 - P^2\dot{m}_2 - \ \ldots) - P\mu(\dot{m}_0 - P\dot{m}_1 - P^2\dot{m}_2 - \ \ldots) = 0$

Now collecting the coefficient of the resultant equation based on powers of P-terms, we get;

$P^0 : \dot{m}_0 = 0$ - - - - - - - -- (4.24)

$P^1 : \dot{m}_1 - \omega\dot{m}_0 - \mu\dot{m}_0 = 0$ - - - - - - - (4.25)

$P^2 : \dot{m}_2 - (\omega - \mu)\dot{m}_1 = 0$ - - - - - - - (4.26)

Also, applying Homotopy perturbation method (HPM) to (4.5), we get

$(1-P)\frac{dI_r}{dt} - P\left[\frac{dI_r}{dt} - \gamma I - \theta S - \mu I_r\right] = 0$

Removing bracket, we get;

$\frac{dI_r}{dt} - P\gamma I - P\theta S - P\mu I_r = 0$

Substituting (4.7) to (4.11) into equation (4.27), we expand to get

$\dot{n}_0 - P\dot{n}_1 - P^2\dot{n}_2 - \ \ldots P\gamma(y_0 - Py_1 - P^2y_2 - \ \ldots) - P\theta(x_0 - Px - P^2x_2 - \ \ldots) - P\mu(n_0 - Pn_1 - P^2n_2 - \ \ldots) = 0$

Now collecting the coefficient of the resultant equation based on powers of P – terms, we get

$P^0 : \dot{n}_0 = 0$ - - - - - - - - (4.28)

$P^1 : \dot{n}_1 - \gamma y_0 - \theta x_0 - \mu n_0 = 0$ - - -- - - (4.29)

$P^2 : \dot{n}_2 - \gamma y_1 - \theta x_1 - \mu n_1 = 0$ - - - - - (4.30)

From (4.12)

$\dot{x} = 0$

Integrating, we have

$\frac{dx_0}{dt} = 0 \Rightarrow dx_0 = 0 \Rightarrow \int dx_0 = 0 \Rightarrow x_0 = A$

Initial conditions, we have $x_o(0) = S_0$:- $S_o = x_o$ - - (4.31) considering equation (4.16)

$y_0 = 0$

$\frac{dy_0}{dt} - 0, \rightarrow dy_0 - 0 \rightarrow \int dy_0 - 0, \rightarrow y_0 - B$

Initial conditions, we have, :- $y_0 = I_0$ - -- - (4.32)

For equation (4.20)

$\dot{z_0} = 0$

Integrating, we have

$\frac{dz_0}{dt} = 0, \Rightarrow dz_0 = 0 \Rightarrow \int dz_0 = 0, \Rightarrow z_0 = C$

Initial conditions, we have,

:- $z_0 = R_0$ - - - - - - - - (4.33)

For equation (4.24)

$m_0 = 0$ (integrating, we have) $\int dm_0 = 0, \Rightarrow m_0 = D$

Initial condition, we have,

$\therefore m_o = S_v(o)$ - - - - - -- (4.34)

Considering equation (4.28)

$\dot{n_0} = 0$

upon integration,

$\frac{dn_0}{dt} = 0, \Rightarrow dn_0 = 0 \Rightarrow \int dn_0 = 0, \Rightarrow n_0 = E$

Applying the initial condition, we get

$\therefore n_o = I_v(o)$ - - - - - - -- - (4.35)

Now, substituting (4.31), (4.32), (4.33), (4.34) and (4.35) into (4.13)

$\dot{x} = -\beta x_0 n_0 - (1-\alpha) z_0 - \theta n_0 - \mu x_0$

$\Rightarrow \dot{x_1} = -\beta S_0 I_{v0} - (1-\alpha) R_0 - \theta I_{v0} - \mu S_0$

Integrating both sides, we have

$\frac{dx_1}{dt} = -S_0(\beta I_{v0} - \mu) - (1-\alpha) R_0 - \theta I_{v0}$

Integrating both sides, we have

$\int dx = (-S_0(\beta I_{v0} - \mu) - (1-\alpha) R_0 - \theta I_{v0}) \int dt$

$X_1 = (-S_0(\beta I_{v0} - \mu) - (1-\alpha) R_0 - \theta I_{v0})\, t - c$

Initial condition, we obtain, $x_1(0) = 0, \Rightarrow c = 0$

$\therefore x_1 = (-S_0(\beta I_{v0} - \mu) - (1-\alpha) R_0 - \theta I_{v0}) t$ - - -(4.36)

Substituting equation (4.31), (4.32), (4.33), (4.34) and (4.35) into (4.17)

$\dot{y}_1 = \beta x_0 y_0 - \gamma n_0 - \alpha z_0 - \mu y_0, \quad \Rightarrow \dot{y}_1 = \beta S_0 I_0 - \gamma I_{v0} - \alpha R_0 - \mu I_0$

$\frac{dy_1}{dt} = \beta S_0 I_0 - \gamma I_{v0} - \alpha R_0 - \mu I_0$

Integrating both sides, we have

$\int dy_1 = (I_0(\beta S_0 - \mu) - \gamma I_{v0} - \alpha R_0) \int dt, \quad y_1 = (I_0(\beta S_0 - \mu) - \gamma I_{v0} - \alpha R_0)\, t - c_1$

Initial condition, we obtain, $y_1(0) = 0, \ \Rightarrow c_1 = 0$

$\therefore y_1 = (I_0(\beta S_0 - \mu) - \gamma I_{v0} - \alpha R_0)\, t$ - - - - -(4.37)

Substituting equation (4.33) into equation (4.21)

$\dot{z}_1 = (\alpha - \mu - (1-\alpha))z_0 , \dot{z}_1 = (\alpha - \mu - (1-\alpha))R_0$

$\frac{dz_1}{dt} = (\alpha - \mu - (1-\alpha))R_0$

Integrating both sides, we get

$\int \mathbf{dz_1} = (\alpha - \mu - (1-\alpha))R_0 \int dt$

$z_1 = R_0(\alpha - \mu - (1-\alpha))t - c_2$

By the initial condition, we obtain, $z_1(0) = 0, \ \Rightarrow c_2$

$\therefore z_1 = R_0(\alpha - \mu - (1-\alpha))t$ - - - -- (4.38)

Substituting equation (4.34) into equation (4.25)

$\dot{m}_1 = -S_{v0}(\omega - \mu), \ \Rightarrow \frac{dm_1}{dt} = -S_{v0}(\omega - \mu)$

Integrating both sides, we get

$\int \mathbf{dm_1} = -S_{v0}(\omega - \mu) \int \mathbf{dt}, \ m_1 = -S_{v0}(\omega - \mu)t - c_3$

By the initial condition, we obtain, $m_1(0) = 0, \ \Rightarrow c_3 = 0$

$\therefore m_1 = -S_{v0}(\omega - \mu)t$ - - - - - - (4.39)

Substituting equation (4.31), (4.32) and (4.35) into equation (4.29)

$\dot{n}_1 = \gamma y_0 - \theta x_0 - \mu n_0, \ \Rightarrow \dot{n}_1 = \gamma I_0 - \theta S_0 - \mu I_{v0}$

$\frac{dn_1}{dt} = \gamma I_0 - \theta S_0 - \mu I_{v0}$

Integrating both sides, we get

$\int \mathbf{dn_1} = (\gamma I_0 - \theta S_0 - \mu I_{v0}) \int \mathbf{dt}, \ \Rightarrow \ n_1 = (\gamma I_0 - \theta S_0 - \mu I_{v0})t - c_4$

Applying initial condition, we now have $n_1(0) = 0, \ \Rightarrow \ c_4 = 0$

$\therefore n_1 = (\gamma I_0 - \theta S_0 - \mu I_{v0})t$ - - - - (4.40)

In the same fashion, we substitute equation (4.31), (4.35), (4.36), (4.38) and (4.40) into equation (4.14)

$\dot{x}_2 = -\beta(x_0 n_1 - x_1 n_0) - (1-\alpha)z_1 - \theta n_1$

$\dot{x}_2 = -\beta S_0 n_1 - \beta x_1 I_{v0} - (1-\alpha)z_1 - \theta n_1, \ \Rightarrow \dot{x}_2 = n_1(\theta - \beta S_0) - \beta x_1 I_{v0} - (1-\alpha)z_1$

$\dot{x}_2 - (\theta - \beta S_0)(\gamma I_0 t - \theta S_0 t - \mu I_{v0} t) - \beta I_{v0}(-S_0 t(\beta I_{v0} - \mu) - (1-\alpha)R_0 t - \theta I_{v0} t) - (1 - \alpha)(R_0 \alpha t - \quad R_0 \mu t - (1-\alpha)R_0 t)$

$\dot{x}_2 = \theta\gamma I_0 t - \theta^2 S_0 t - \theta\mu I_{v0} t - \beta S_0\gamma I_0 t - \beta\theta S_0{}^2 t - \beta S_0 \mu I_{v0} t - \beta^2 S_0 I_{v0}{}^2 t - \beta S_0 I_{v0}\mu t - (1 - \alpha)\beta I_{v0} R_0 t - \beta\theta I_{v0}{}^2 t - (1-\alpha)R_0\alpha t - (1-\alpha)R_0\mu t - (1-\alpha)^2 R_0 t$

$\dot{x}_2 = \theta\gamma I_0 t - \theta^2 S_0 t - \theta\mu I_{v0} t - \beta S_0\gamma I_0 t - \beta\theta S_0{}^2 t - 2\beta S_0 \mu I_{v0} t - \beta^2 S_0 I_{v0}{}^2 t - (1 - \alpha)\beta I_{v0} R_0 t - \beta\theta I_{v0}{}^2 t - (1-\alpha)R_0\alpha t - (1-\alpha)R_0\mu t - (1-\alpha)^2 R_0 t$

$\Rightarrow$

$\frac{dx_2}{dt} = \left(S_0\left(\theta(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)\right) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})\right) - R_0(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1))t$

Integrating both sides, we get

$\Rightarrow$

$\int dx_2 - \left(S_0\left(\theta(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)\right) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})\right) - R_0(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1)) \int t dt$

$x_2 = \left(S_0\left(\theta(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)\right) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})\right) - R_0\left(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1)\right)t^2/2 - c_5$

Applying initial condition, $x_2(0) = 0, \Rightarrow c_5 = 0$

$\therefore\ x_2 = \left(S_0\left(\theta(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)\right) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})\right) - R_0\left(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1)\right)t^2/2$

Substituting equation (4.31), (4.36) and (4.41) into equation (4.7)

We get series

$S(t) = S_0 - p(S_0(\beta I_{v0} - \mu) - (1-\alpha)R_0 - \theta I_{v0})t - p^2 S_0(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})) - R_0(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1))t^2/2$

Setting p=1, the solution of equation (4.7) becomes

$S(t) = \lim_{x\to 1} x = x_0 - x_1 - x_2 - \ldots$

Hence,

$$S(t) = S_0 - (S_0(\beta I_{v0} - \mu) - (1-\alpha)R_0 - \theta I_{v0})t - S_0(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})) - R_0(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1))t^2/2 \quad (4.42)$$

Substituting equation (4.31), (4.32), (4.36), (4.37) and (4.38) into equation (4.18)

$\dot{y}_2 = \beta(x_0 y_1 - x_1 y_0) - \gamma y_1 - \alpha z_1 - \mu y_1$

$\dot{y}_2 = (\beta S_0 - \gamma - \mu)y_1 - \beta x_1 I_0 - \alpha z_1$

$\dot{y}_2 = \left((\beta S_0 - \gamma - \mu)(I_0(\beta S_0 - \mu) - \gamma I_{v0} - \alpha R_0) - \beta I_0(- S_0(\beta I_{v0} - 1) - (1-\alpha)R_0 - \theta I_{v0}\right) - \alpha(R_0(2\alpha - \mu - \beta I_{v0} - 1)))$

$$\dot{y}_2 = \Big(\big(I_0\beta^2{S_0}^2 - I_0\beta S_0\mu - \beta S_0\gamma I_{v0} - \beta S_0\alpha R_0 - I_0\beta S_0\gamma - I_0\mu\gamma - \gamma^2 I_{v0} - \alpha\gamma R_0 - I_0\beta S_0\mu - I_0\mu^2 - \gamma I_{v0}\mu - \alpha R_0\mu\big) - \big(-I_0\beta^2 S_0 I_{v0} - I_0\beta S_0\mu - I_0\beta R_0(1-\alpha)I_0\beta\theta I_{v0}\big) - 2R_0\alpha^2 - R_0\alpha^2\mu - \alpha R_0\Big)t$$

$$\dot{y}_2 = \Big(I_0\big(\beta^2{S_0}^2 - \beta S_0\mu - \beta S_0\gamma - \mu\gamma -- \beta S_0\mu - \mu^2 - \beta^2 S_0 I_{v0} - \beta S_0\mu - \beta R_0(1-\alpha) - \beta\theta I_{v0}\big) - \big(-\beta S_0\gamma I_{v0} - \beta S_0\alpha R_0 - \gamma^2 I_{v0} - \alpha\gamma R_0 - \gamma I_{v0}\mu - \alpha R_0\mu - 2R_0\alpha^2 - R_0\alpha^2\mu - \alpha R_0\big)\Big)t$$

$$\dot{y}_2 = \big(I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu - \mu(\gamma-\mu) - \beta(R_0(1-\alpha) - \theta I_{v0}) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma \quad \mu) \quad \alpha R_0(\gamma \quad 2\alpha \quad 1)\big)t$$

$$\int dy_2 = \big(I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu - \mu(\gamma-\mu) - \beta(R_0(1-\alpha) - \theta I_{v0}) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma-\mu) - \alpha R_0(\gamma - 2\alpha - 1)\big)\int t\,dt$$

$$y_2 = \big(I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu - \mu(\gamma-\mu) - \beta(R_0(1-\alpha) - \theta I_{v0}) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma-\mu) - \alpha R_0(\gamma - 2\alpha - 1)\big)t^2/2 - c_6$$

Applying initial condition,

$$y_2(0) = 0, \ \Rightarrow c_6 = 0$$

$$\therefore y_2 = \big(I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu - \mu(\gamma-\mu) - \beta(R_0(1-\alpha) - \theta I_{v0}) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma-\mu) - \alpha R_0(\gamma - 2\alpha - 1)\big)t^2/2$$

\- - - -(4.43)

Substituting equation (4.32), (4.37) and (4.43) into equation (4.8)
Setting p = 1, the solution of equation (4.8) becomes,

$$I(t) = \lim_{p\to 1} y = y_0 - y_1 - y_2 - \ \ldots$$

Hence,

$$I(t) = I_0 - (I_0(\beta(S_0 - \mu) - \gamma I_{v0} - \alpha R_0)t - (I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu) - \mu(\gamma-\mu) - \beta\big(R_0(1-\alpha) - \theta I_{v0}\big) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma-\mu) - \alpha R_0(\gamma - 2\alpha - 1))t^2/2$$

-(4.44)

Substituting equation (4.38) into equation (4.22)

$$\dot{Z}_2 = (\alpha - \mu - (1-\alpha))z_1$$

$$\dot{Z}_2 = \big(\alpha - \mu - (1-\alpha)\big).R_0\big(\alpha - \mu - (1-\alpha)\big)t$$

$$\dot{Z}_2 = R_0(\alpha - \mu - (1-\alpha))^2 t$$

$$\frac{dz_2}{dt} = R_0(\alpha - \mu - (1-\alpha))^2 t$$

Integrating both sides we get

$$\int dz_2 = R_0(\alpha - \mu - (1-\alpha))^2 \int t\,dt$$

$$Z_2 = R_0(\alpha - \mu - (1-\alpha))^2\, t^2/2 \ - c_7$$

Applying initial condition, we have $z_2(0) = 0, \implies c_7 = 0$

$$\therefore Z_2 = R_0(\alpha - \mu - (1-\alpha))^2 t^2/2 \quad - \quad - \quad - \quad - \quad (4.45)$$

Substituting equation (4.33), (4.38) and (4.43) into equation (4.9)

$$R(t) = R_0 - pR_0(\alpha - \mu - (1-\alpha))t - p^2R_0(\alpha - \mu - (1-\alpha))^2 t^2/2$$

Setting $p = 1$, the solution of equation (4.9) becomes

$$R(t) = \lim_{p \to 1} z = z_0 - z_1 - z_2 - \ . \ . \ .$$

Hence,

$$R(t) = R_0 - R_0(\alpha - \mu - (1-\alpha))t - R_0(\alpha - \mu - (1-\alpha))^2 t^2/2 \quad - \quad - \quad (4.46)$$

Substituting equation (4.39) into equation (4.26)

$$\dot{m}_2 = -(\omega - \mu)m_1$$

$$\dot{m}_2 = -(\omega - \mu)(-S_{v0}(\omega - \mu))t = S_{v0}(\omega - \mu)^2 t$$

$$\frac{dm_2}{dt} = S_{v0}(\omega - \mu)^2 t$$

$$\int dm_2 = S_{v0}(\omega - \mu)^2 \int dt$$

$$m_2 = S_{v0}(\omega - \mu)^2 t^2/2 - c_8$$

Applying Initial condition, $m_2(0), \implies c_8 = 0$

Hence,

$$m_2 = S_{v0}(\omega - \mu)^2 t^2/2 \quad - \quad - \quad - \quad - \quad - \quad - \quad (4.47)$$

Substituting equation (4.34), (4.39) and (4.47) into equation (4.10)

$$S_v(t) = S_{v0} - pS_{v0}(\omega - \mu)t - p^2 S_{v0}(\omega - \mu)^2 t^2/2$$

Setting $p = 1$, then solution of equation (4.10) becomes

$$S_v(t) = \lim_{p \to 1} m = m_0 - m_1 - m_2 - \ . \ . \ .$$

$$S_v(t) = S_{v0} - S_{v0}(\omega - \mu)t - S_{v0}(\omega - \mu)^2 t^2/2 \quad - \quad - \quad - \quad - \quad (4.48)$$

Lastly, we substitute equation (4.36), (4.37), (4.40) into equation (4.30)

$$\dot{n}_2 = \gamma y_1 - \theta x_1 - \mu n_1$$

$$\dot{n}_2 = \gamma(I_0(\beta S_0 - \mu) - \gamma I_{v0} - \alpha I_{v0} - \alpha R_0)t - \theta(-S_0(\beta I_{v0} - \mu) - (1-\alpha)R_0 - \theta I_{v0})t - \mu(\gamma I_0 - \theta S_0 - \mu I_{v0})t$$

$$\dot{n}_2 = (\gamma(I_0\beta S_0 \quad I_0\mu \quad \gamma I_{v0} \quad \alpha I_{v0} \quad \alpha R_0) \quad \theta(\quad I_{v0}\beta S_0 \quad S_0\mu \quad (1 \quad \alpha)R_0 \quad \theta I_{v0}) \quad \mu(\gamma I_0 \quad \theta S_0 - \quad \mu I_{v0}))t$$

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$n_2 = (I_{v0}(\gamma\alpha - \gamma^2 - \beta S_0\theta - \theta^2 - \mu^2) - I_0(\beta S_0\gamma - \gamma\mu - \gamma\mu) - (\alpha\gamma R_0 - S_0\mu\theta - (1 - R_0)R_0\theta - S_0\theta\mu))t$

$n_2 = (I_{v0}\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))t$

$\int \frac{dn_2}{dt} = (I_{v0}\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu)) \int t\,dt$

$n_2 = (I_{v0}\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))\, t^2/2 - c_3$

Applying the initial condition, $n_2(0) = 0$, $\Longrightarrow c_9 = 0$

∴

$n_2 = (I_{v0}\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))\, t^2/2$

\- - - - (4.49)

Substituting equation (4.35), (4.40) and (4.49) into equation (4.11)

$I_v(t) = I_{v0} - p(\gamma I_0 - \theta S_0 - \mu I_{v0})t - p^2(I_{v0}(\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))\, t^2/2$

Setting $p = 1$, the solution of the equation (4.11) becomes

$I_{vt}(t) = \lim_{p\to 1} n = n_0 - n_1 - n_2 - \ldots$

Hence,

$I_v(t) = I_{v0} - (\gamma I_0 - \theta S_0 - \mu I_{v0})t - (I_{v0}(\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))\, t^2/2$

\- - - - - -(4.50)

The general solutions of equations (4.1), (4.2), (4.3), (4.4) and (4.5) are equations (4.42), (4.44), (4.46), (4.48) and (4.50) respectively, that is,

$S(t) = S_0 - (S_0(\beta I_{v0} - \mu) - (1 - \alpha)R_0 - \theta I_{v0})t - S_0(\beta S_0 - \theta) - \beta I_{v0}(\beta I_{v0} - 2\mu - \gamma)) - \theta(\gamma I_0 - I_{v0}(\mu - \beta I_{v0})) - R_0(1 - \alpha(2\alpha - \mu - \beta I_{v0} - 1))t^2/2$

$I(t) = I_0 - (I_0(\beta(S_0 - \mu) - \gamma I_{v0} - \alpha R_0)t - (I_0(\beta^2 S_0(S_0 - I_{v0}) - \beta S_0\mu) - \mu(\gamma - \mu) - \beta(R_0(1 - \alpha) - \theta I_{v0}) - \beta S_0(\alpha R_0 - \gamma I_{v0}) - \gamma I_{v0}(\gamma - \mu) - \alpha R_0(\gamma - 2\alpha - 1))t^2/2$

$R(t) = R_0 - R_0(\alpha - \mu - (1 - \alpha))t - R_0(\alpha - \mu - (1 - \alpha))^2\, t^2/2$

$S_v(t) = S_{v0} - S_{v0}(\omega - \mu)t - S_{v0}(\omega - \mu)^2\, t^2/2$

$I_v(t) = I_{v0} - (\gamma I_0 - \theta S_0 - \mu I_{v0})t - (I_{v0}(\gamma(\alpha - \gamma) - \theta(\beta S_0 - \theta) - \mu^2) - I_0\gamma(\beta S_0 - 2\mu) - (\alpha\gamma R_0 - (1 - R_0)R_0\theta - 2S_0\theta\mu))\, t^2/2$

RESULTS AND DISCUSSION

The model at Equilibrium State

At equilibrium $\frac{ds}{dt} = \frac{dI}{dt} = \frac{dR}{dt} = 0$

Let $S = x,\ I = y,\ r = z,\ S_v = m$ and $I_v = n$

Equation (5.1), (5.2) and (5.3) becomes

$-\beta xn - \mu x - (1-\alpha)z - \theta n = 0$ - - -- - - - (5.1)

$\beta xy - \mu y - \gamma n - \alpha z = 0$ - - - - - - (5.2)

$\alpha z - \mu z - (1-\alpha)z = 0$ - - - - - - (5.3)

From equation (5.3) $z = 0$ - - - - - (5.4)

Put equation (5.4) into equation (5.1)

$-\beta xn - \mu x - \theta n = 0$

$x(-\beta n - \mu) - \theta n = 0$

$x(-\beta n - \mu) = -\theta n$

$\therefore\ x = \frac{\delta n}{\beta x - \mu}$ - - - -- - - - - (5.5)

From equation (5.2)

$\beta xy - \mu y - \gamma n - \alpha z = 0$

$\beta xy - \mu y - \gamma n = 0$

$y = \frac{\gamma n}{\beta x - \mu}$

but, $x = \frac{\delta n}{\beta x - \mu}$

$\Longrightarrow \ = \frac{\gamma n}{\beta\left(\frac{\delta n}{\beta x + \mu}\right) - \mu}$

$y = \frac{\delta n(\beta n - \mu)}{\beta\delta n - \mu(\beta n - \mu)}$ - - - - - - - - (5.6)

$x = \frac{\delta n}{\beta x - \mu}$, $\quad y = \frac{\delta n(\beta n - \mu)}{\beta\delta n - \mu(\beta n - \mu)}$ and $z = 0$

The Disease Free Equilibrium

The absence of infection, where $y = 0$ is known as disease free equilibrium or zero equilibrium.

Hence, we substitute $y = 0$ into equation (5.2)

$\beta xy - \mu y - \gamma n - \alpha z = 0$

$\beta x(0) - \mu(0) - \gamma n - \alpha z = 0$ - - - - - - (5.7)

$\Longrightarrow x = 0,\ y = 0,\ z = 0$

$x = y = z = 0$, we have the disease free equilibrium as

$(x, y, z) = (0, 0, 0)$ - - - - - - - - (5.8)

The Endemic Equilibrium State

The presence of infection, where $y \neq 0$ is known as endemic or non-zero

$$(x, y, z) = \left(x = \frac{\delta n}{\beta x - \mu}, \quad y = \frac{\delta n(\beta n - \mu)}{\beta \delta n - \mu(\beta n - \mu)}, \quad z = 0\right) \quad - \quad - \quad - \quad - (5.9)$$

Stability Analysis of the Disease Free Equilibrium (DFE)

Given the system of equation, the model at equilibrium state as

$$-\beta x n - \mu x - (1-\alpha)z - \theta n = 0$$

$$\beta x y - \mu y - \gamma n - \alpha z = 0$$

$$\alpha z - \mu z - (1-\alpha)z = 0$$

Consider the characteristics vector A, expressed

$$\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{bmatrix} -\beta n & -\mu & 0 \\ \beta x & -\mu & 0 \\ \alpha & -\mu & -(1-\alpha) \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The matrix A of the system of equation is given by

$$A = \begin{vmatrix} -\beta n & -\mu & 0 \\ \beta x & -\mu & 0 \\ \alpha & -\mu & -(1-\alpha) \end{vmatrix} = 0$$

The characteristic determinant of A is given by

$$\det|A - \lambda I| \det A = \begin{vmatrix} (-\beta n - \lambda) & -\mu & 0 \\ \beta x & -(\mu - \lambda) & 0 \\ \alpha & -\mu & -(1 - \alpha - \lambda) \end{vmatrix} = 0$$

$$(-\beta n - \lambda)\left((\mu - \lambda)(1 - \alpha - \lambda)\right) + \mu\left(-\beta x(1 - \alpha - \lambda)\right) + 0$$

$$\Rightarrow \left((-\beta n - \lambda)(\mu - \lambda)(1 - \alpha - \lambda)\right) = 0 \quad - \quad - \quad - \quad - \quad -$$

(5.10)

Recall from equation (5.8) that DFE is given as

(x,y,z) = (0, 0, 0)

It follows that,

$$\text{Det } |A - \lambda I| = \det \begin{vmatrix} (-\beta n - \lambda) & -\mu & 0 \\ \beta x & -(\mu - \lambda) & 0 \\ \alpha & -\mu & -(1 - \alpha - \lambda) \end{vmatrix} = 0$$

$$\Rightarrow \left((-\beta n - \lambda)(\mu - \lambda)(1 - \alpha - \lambda)\right) = 0 \quad - \quad - \quad - \quad - \quad -$$

(5.11)

Either $-\beta n - \lambda = 0$ or $\mu + \lambda = 0$ $1 - \alpha + \lambda = 0$

$\Rightarrow \lambda = \frac{\beta}{}n$ or $\lambda_2 = -\mu$ or $\lambda_3 = \frac{\alpha}{} - 1$ - - - - - (5.12)

From equation (5.12)

$\lambda_2 < 0$

$\lambda_1 > 0$

$\lambda_3 < 0$ if $\frac{\alpha}{} < 1$

$\lambda_3 > 0$ if $\frac{\alpha}{} > 1$

Hence, the DFE will be stable if $\frac{\alpha}{} < 1$ and unstable if $\frac{\alpha}{} > 1$

Stability Analysis of the Endemic Equilibrium (EE)

At non-zero equilibrium we obtain,

$(x,y,z) = \left(\frac{\delta n}{\beta x - \mu}, \frac{\delta n(\beta n - \mu)}{\beta \delta n - \mu(\beta n - \mu)}, 0\right)$

From equation (5.10) we expand to have

$\lambda^3 + \frac{\beta n}{}\lambda^3 + \mu\lambda^2 + \lambda^2 - \frac{\alpha}{}\lambda^2 + \frac{\beta n\lambda\mu}{} + \frac{\beta n\lambda}{} - \frac{\alpha\beta n\lambda}{} + \mu\lambda - \frac{\alpha\lambda\mu}{} + \frac{\beta n\mu}{} - \frac{\alpha\beta n\mu}{} + \frac{\beta x\mu}{} - \frac{\alpha\beta x\mu}{} = 0$

Collecting like terms of λ

$\lambda^3 + \left(\frac{\beta n}{} + \frac{\mu}{} + 1 - \frac{\alpha}{}\right)\lambda^2 + \left(\frac{\beta n\mu}{} + \frac{\beta n}{} + \frac{\beta n\lambda}{} + \frac{\mu}{} - \frac{\alpha\mu}{}\right)\lambda + \left(\frac{\beta n}{} - \frac{\alpha\beta n}{} + \frac{\beta nx}{} - \frac{\alpha\beta nx}{}\right)\mu = 0$ - - - - - - - - (5.13c)

We apply Bellman and Cooke theorem of stability theorem:

$$H(Z) = P(Z, e^z),$$

where $P(z, w)$ is a polynomial with principal term.

Suppose $H(iy), Y \in R$ is separated into real and imaginary parts.

$$H(iy) = F(y) - G(y) \qquad (5.14)$$

If all zeros of H(z) having negative real parts, then zeros of F(y) and G(y) are real, simple and alternate and

$F(0)\dot{G}(0) > 0$ for all $Y \in R$ - - - - - - - - - - (5.15)

Conversely, all zeros of H(z) will be in the left- half plane provided that either of the following conditions is satisfied.

(i) All the zeros of F(y) and G(y) are real simple and alternate and the inequality in (5.15) is satisfied for at least one y.
(ii) All the zeros of F(y) are real and, for each zero, the relation (5.15) is satisfied.

Let (5.13c) be expressed as

$$H(\lambda) = \lambda^3 - (\beta n - \mu - 1 - \alpha)\lambda^2 - (\beta n\mu - \beta n - \beta n\alpha - \mu - \alpha\mu)\lambda - (\beta n - \alpha\beta n - \beta nx - \alpha\beta nx)\mu$$

Setting $\lambda = iw$, we have

$$H(iw) = F(W) - ig(w) \quad - \quad (5.17)$$

Where F(y) and G(y) are the real and imaginary parts of (5.17)

Substituting $\lambda = iw$ into (5.16), we get

$$H(iw) = (iw)^3 - (\beta n - \mu - 1 - \alpha)(iw)^2 - (\beta n\mu - \beta n - \beta n\alpha - \mu - \alpha\mu)(iw) - (\beta n - \alpha\beta n - \beta nx - \alpha\beta nx)\mu$$

$$= -iw^3 - w^2(\beta n - \mu - 1 - \alpha) - (\beta n\mu - \beta n - \beta n\alpha - \mu - \alpha\mu)(iw) - (\beta n - \alpha\beta n - \beta nx - \alpha\beta nx)\mu \quad - \quad (5.18)$$

Separating the real and imaginary parts of (5.18b), we have

$$F(w) = -w^2(\beta n - \mu - 1 - \alpha) \quad - \quad (5.19)$$

$$G(w) = (\beta n\mu - \beta n - \beta n\alpha - \mu - \alpha\mu) - w^2 \quad - \quad (5.20)$$

Differentiating (5.19) and (5.20) with respect to w, we have

$$\acute{F}(w) = -2w(\beta n - \mu - 1 - \alpha) \quad - \quad (5.21)$$

$$\acute{G}(w) = 2w \quad - \quad (5.22)$$

Set $w = 0$

$$\acute{F}(0) = 0 \quad - \quad (5.23)$$

$$\acute{G}(0) = 0 \quad - \quad (5.24)$$

$$F(0) = 0 \quad - \quad (5.25)$$

$$G(0) = 0$$

Applying the initial conditions for stability,

Since $F(0)\acute{G}(0) - \acute{F}(0)G(0) > 0$

Let $k = F(0) \times \acute{G}(0) = 0$

By the above theorem, K > 0 implies stability otherwise instability.

Simulation

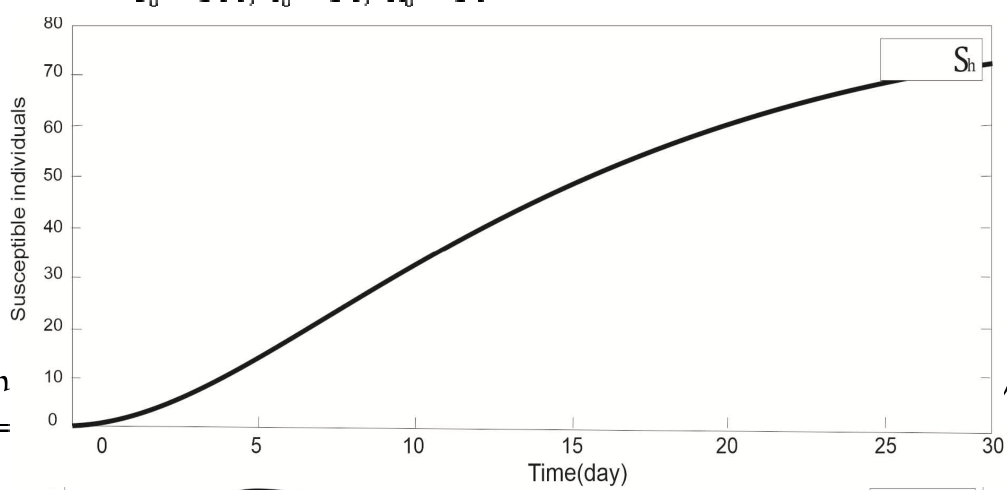
In this section we simulate some standard scenarios to explore the behavior of the model. This can be done in an infinite number of ways, but we have chosen five main scenarios based on the transmission rate. This corresponds in some to real life scenarios where the environmental condition provides different possibilities for the spread of disease.

Graphical Representation of the Model Using MATLAB

Here we now show the graphs generated from the general solutions of our model (that is equations (4.42), (4.44), (4.46), (4.48) and (4.50)). We use hypothetical values to generate tables and graphs

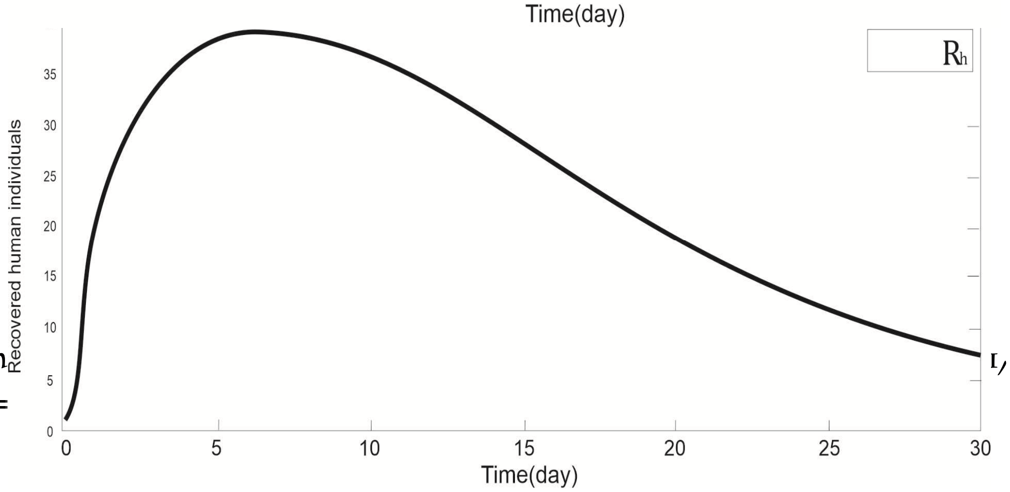
Simulated Results for $\alpha = 0.001$, $\beta = 0.015$, $\gamma = 0.01$, $\mu = 0.1$,

$S_0 = 500,\ I_0 = 50,\ R_0 = 50$



Sim ,

$S_0 =$



Sim 1,

$S_0 =$

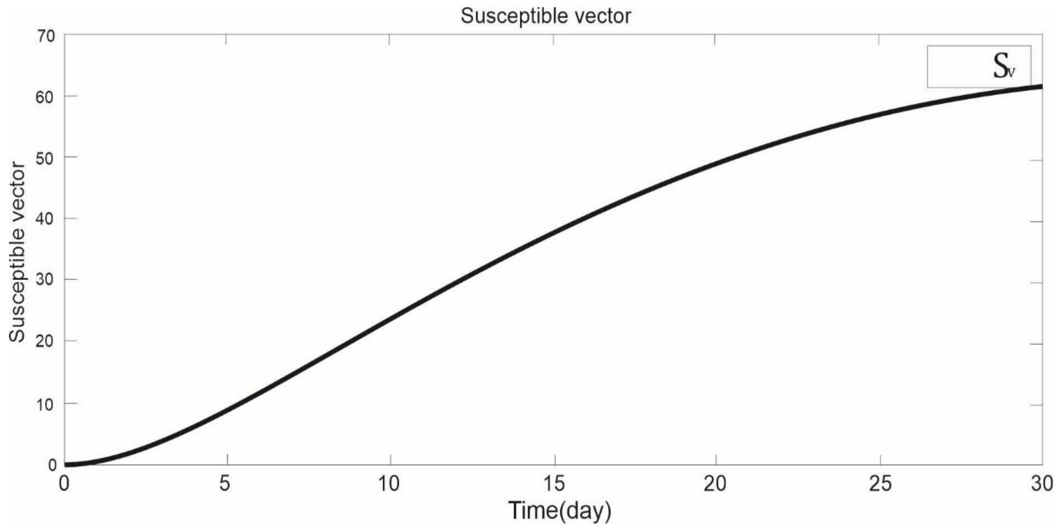


Figure-3

Represent the recovered individuals in the population

Simulated Results for $\alpha = 0.001$, $\beta = 0.015$, $\gamma = 0.01$, $\mu = 0.1$, $S_{v0} = 500$, $I_{v0} = 50$,

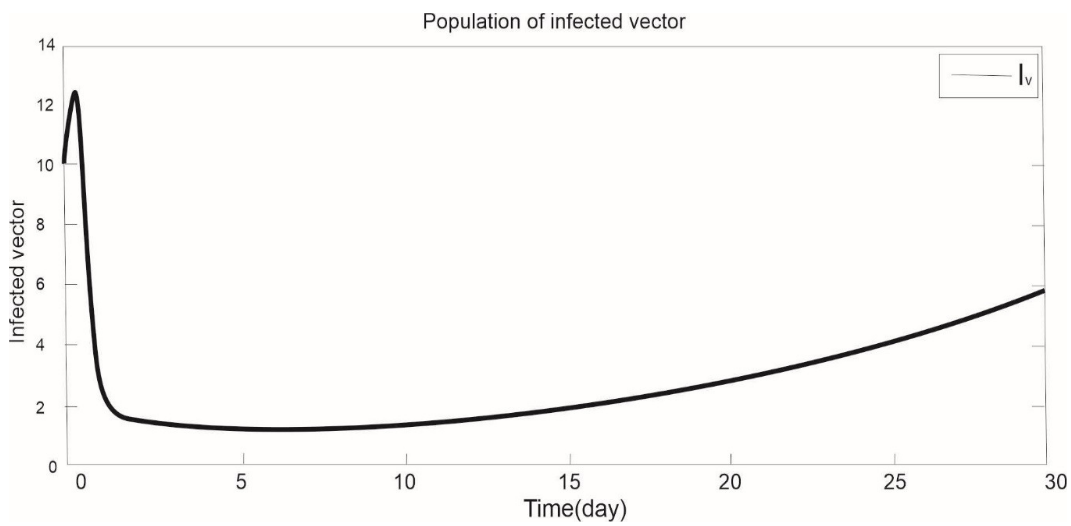
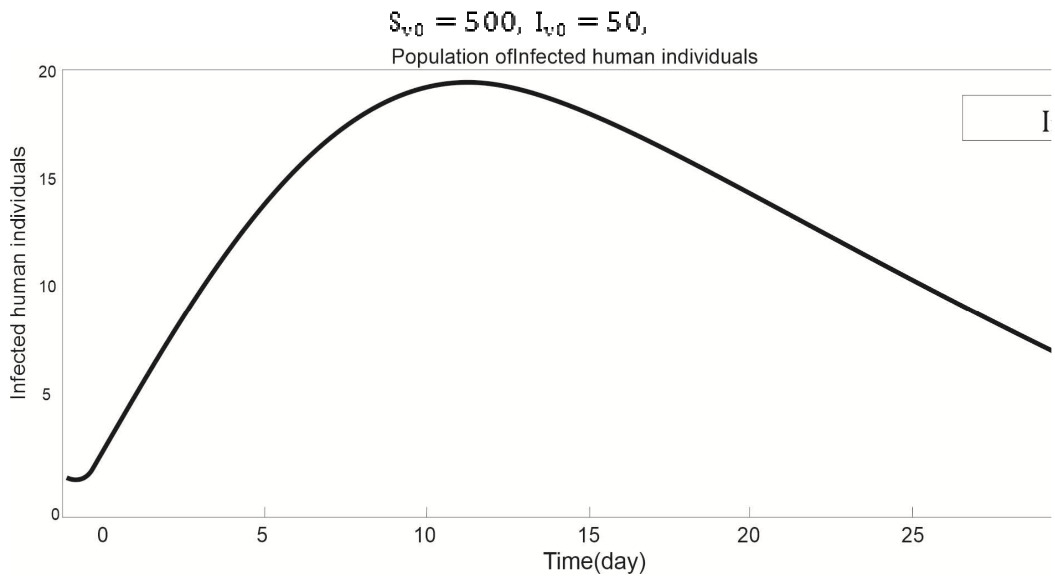


Figure-4

Represent the Susceptible vector in the population

Simulated Results for $\alpha = 0.003$, $\beta = 0.015$, $\gamma = 0.01$, $\mu = 0.1$,



DISCUSSION OF RESULTS

We notice from eqn(5.12) that the disease free equilibrium (DFE) will be stable if $\alpha<1$. Nevertheless, since the Eigen values are not all negative, we conclude that the zero equilibrium state is unstable. Applying Bellman and Cooke's theorem to verify that the stability of the endemic equilibrium, we discovered that our non-zero equilibrium could not meet the condition for stability according bellman and cooke, Hence the result of our analysis shows that the non-zero equilibrium is unstable. The implication of instability of the system is that, the population cannot withstand widespread occurrence of the disease.

CONCLUSION

The disease free equilibrium will be stable if $\alpha<1$. This means that the population is sustainable. The instability of the non-zero equilibrium state that in case of any disease outbreak, the population is over powered. We deduced from the graphical justification that no matter the transmission rate, if there is a concerted effort in combating the disease outbreak the population is sustained.

REFERENCES

Alemu A., Abebe G., Tsegaye, W. and Golassa L, [2011], "Climate variables and malaria Transmission Dynamics in Jimma Town, South West Ethiopia" parasite and vectors 4[1]:30.doi: 10.1186/1756-3305-4-30.

Alonso D., Bouma M.J., and Pascal [2011], "Epidemic malaria and warmer Temperatures in recent decades in East African Highland". Proceedings of the Royal society B: Biological Sciences 278 [1712] 1661-1669

Brauer F., Nohel J.A., Qualitative Theory of Ordinary Differential Equations Dover, 1969.

Carter R.[2002]. "Spatial simulation of malaria transmission and it control by malaria transmission blocking vaccination". International Journal for Parasitology, volume 32, pp 1624.

Gomez- Elipe A., Otero A., Van Harp M., and Aguirre-Jamie A [2003]. "Forecasting Malaria incidence based on monthly case reports and environmental factors 2003-2012". Biomed central volume 6, pp. 129

Kalkami M.A., Descrochers RE , and Kerr J.T [2010]. "High-Resolution Niche malaria vectors in northern Tanzania: "A new Capacity to predict malaria Risk". Plos ONE 5[2]: e9396.doi:10:1371/ journal. pone, 009396.

Li- j., [2008], "A model with partial immunity in humans". Maths Biosciences and engineerimg Vol.5, No. 4, pp 789-801.

Li- J., (2008) "A model with partial immunity in humans" . Maths Bio. Malaria Policy Advisory Committee Meeting 13-15 March 2013, WHO HQ sessions 9.

Mbah G.C.E. Introduction to mathematical modeling Manual.

Ndegna B.A.,Simbauni J.A .., Mbugi J.P ., Githeko A.k, and Fillinger U., (2011).

Productivity of malaria vectors from different Habitat Types in the Western Kenya Highlands". Plos ONE 6(4): e19473 .doi.10. 1371/journal .pone, 0019473.

Ngwa G.A ,Shu W.S., A mathematical model for endemic malaria with variable human and mosquito populations. Mathematical and computer modeling, 32(7) (2000),747-756.

Roll Back Malaria. What is Malaria? http://www.roll back malaria .org /cme-upload /0/00015/372/RB Infosheet -1,pdf. (2010-05-10).

Sanchez D.A., Ordinary Differential Equations and stability theory: An introduction W.H ., Freeman and Company : 1768.

Wagala C.L., Waitumbi J.Zhou G.and Githeko A.K (2011) "Identification of Malaria Transmission and epidemic Hotspots in Western Kenya Highlands. Parasites and vectors 4.81.doi:10.1186/1756-3305-4-81.