International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 4, December 2018
http://www.casirmediapublishing.com

Comparing the Level of Stability between Standard (Single-Step) and Multi-Step Methods in Numerical Ordinary Differential Equations

Adetunji, A. B. & Adeleke, I.A.
Department of Computer Science and Engineering,
Lautech, Ogbomoso and Computer Science Department, EACOED, Oyo
abadetunji@lautech.edu.ng and adeleke_israel@yahoo.com

ABSTRACT
In solving ordinary differential equation (ODE) numerically, there are series of techniques to be employed but the choice of suitable technique to determine their level of stability is a great problem. This research evaluated Runge-Kutta of fourth order as standard method and Adams-Moulton of fourth order as multistep method with two sample problems of ordinary differential equations to compare their performances. The two problems were solved with the two numerical algorithms selected and implemented using C++ programming language as ten iterations on each sample problem were carried out to determine their stability. The results obtained were presented in form of tables which ranged between -1.000000 to 1.9524898 for Adams method while Runges values ranged between -1.000000 to -0.3639740 and their corresponding values ranged between 4.0000000 to 5.7797685 and 4.0000000 to 5.8775982 respectively. The error differences in relation to the exact-solutions were calculated such that Adam's method ranged between 5.7E-07 to 2.56E-06 while that of Runge's method ranged between 5.7E-07 to 2.3165 their corresponding error differences ranged between -6.12E-03 to -5.3E-06 and -6.12E-03 to -9.7835E-02 respectively. Graphs were used to present the error differences to clearly see the level of stability of the two numerical algorithms under consideration. The result showed that Adams-Moulton technique is more stable if not absolutely stable than its counterpart.
Keyword: ODE, stability, multistep, standard-method and exact-solution

INTRODUCTION

Numerical analysis, according to Kendall (2007) is the area of mathematics and computer science that creates analyses and implements algorithms for solving numerically the problem that originate generally from real-world application that involve variables which vary continuously. In numerical analysis which is a concept referring to the sensitivity of the solution of a given problem to small changes in the data or the given parameter of the problem, numerical stability is an important notion. For an algorithm to be numerically stable, according to Kendall (2007), it means if an error whatever its cause, does not grow to be much larger during the calculation. In other words, any numerical method is said to be stable if it produces bounded solution which imitates the exact solution (Grewal, 2007). Therefore, according to Grewal (2007), if a method is stable for all values of the parameter, it is said to be absolutely or unconditionally stable but if it is stable for some values of the parameter, it is conditionally stable. The stability of any method is at times affected by the number of the significant digits the machine keeps on, if we use a machine that keeps on the first four floating-point digits. Siddiqi and Manchanda (2006) explained that a differential equation is said to be an ordinary differential equation (ODE) if it contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable. That is, ODE exists if the unknown function y depends on only one independent variable x such as written below

International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 4, December 2018
http://www.casirmediapublishing.com

$$\frac{dy}{dx} = 5x + 3$$

$$e^{y}\frac{d^2y}{dx^2} + 2\frac{(dy)^2}{dx} = 1$$

In any ordinary differential equation, the concern of many numerical analysts is to determine the convergence rate, order of consistency and the level of stability by trying to compare the approximated values that may be gotten when one applies any numerical method to solve numerical problems. According to Cheney and Kincaid (1985), although many methods exist for obtaining analytical solutions of differential equations, they are primarily limited to special differential equations that is why many numerical analysts have carried out series of researches by employing different numerical techniques in solving numerical equations in order to discover the most appropriate technique among those numerous methods. This has triggered many researchers among which includes Butcher (2003) who used Euler method and Runge–Kutta method of fourth order to solve first order differential equation. John (1995) designed an algorithm to numerically solve differential equations with the application of fourth order Runge-Kutta method and Adams-Moulton algorithms of fourth order (predictor – corrector strategy) to approximate the solution of the initial value problem (IVP) dy/dx = f(x, y) with y(a) =y over [a,b] mainly to determine the reliability, applicability and accuracy of these numerical methods in solving first order differential equations. Agam and Odion (2012) employed perturbed Runge-Kutta method for solution of 2^{nd} order differential equations while Atajoromavwo and Onavwie (2012) employed C++ programming language to solve first order differential equation with Runge-Kutta method. Gilberto (2010) used fourth order Runge-Kutta method for solving second order ODE by reducing second order ODE into a system of two first orders by using variable substitution to form a system of ODE.

This research paper focused on comparing the level of stability of numerical techniques and two techniques were employed for solving ordinary differential equations of second order to achieve this which include Runge-Kutta, and Adams-Moulton method both of fourth order when they are subjected to the same step-size and iterations. Runge-Kutta is standing for single step method which Steven and Raymond (2010) expressed as a method which utilize information at a single point x_i to predict a value of the dependent variable y_{i+1} at a future x_{i+1} and Adams-Moulton stands for multistep method which based on the insight that once the computation has begun, valuable information from previous point is at our command.

.

RESEARCH METHOD

Two sample equations of ordinary differential equations (second-order) were selected and solved with Runge-Kutta algorithm of fourth order, and Adams-Moulton algorithm which then coded in C++ programming language after reducing the selected equations into systems of first order. The results (approximated values) obtained from the numerical algorithms were presented in tabular form. Each of the obtained results was subtracted from the exact solutions to determine their error differences. These error differences are then presented in form of table and graphically to discover the level of stability of the two numerical techniques under study.

Runge-Kutta Method of Fourth Order.

Runge –Kutta method of fourth order is the most popular order because it has infinity number of variables which according to Cheney and Kincaid (1985) was designed to imitate Taylor's series method without requiring analytical differentiation of the original differential equation. Also, this Runge-Kutta method achieves the accuracy of a Taylor's series approach without requiring the calculation of higher derivatives (Steven and Raymond, 2010). Therefore, Grewal (2007) sees this Runge-Kutta method of fourth order as the most commonly used which is often referred to as Runge's method. The formula is given as

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4) \qquad 1$$

where

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + 1/2h, y_n + 1/2k_1)$$
$$k_3 = hf(x_n + 1/2h, y_n + 1/2k_2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

the formulas for the fourth order Runge-Kutta algorithm after the conversion of the equation to systems of first order are:

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4) \qquad 2$$
$$z_{n+1} = z_n + 1/6(l_1 + 2l_2 + 2l_3 + l_4) \qquad 3$$

where

$$k_1 = hf(x_n, y_n, z_n)$$
$$l_1 = hg(x_n, y_n, z_n)$$
$$k_2 = hf(x_n + 1/2h, y_n + 1/2k_1, z_n + 1/2l_1)$$
$$l_2 = hg(x_n + 1/2h, y_n + 1/2k_1, z_n + 1/2l_1)$$
$$k_3 = hf(x_n + 1/2h, y_n + 1/2k2, z_n + 1/2l_2)$$
$$l_3 = hg(x_n + 1/2h, y_n + 1/2k_2, z_n + 1/2l_2)$$
$$k_4 = hf(x_n + h, y_n + k_3, z_n + l_3)$$
$$l_4 = hg(x_n + h, y_n + k_3, z_n + l_3)$$

Given starting values x_o, y_o, these can be used to find x_1, y_1 and then x_1, y_1 to find x_2, y_2 and so on.

Adams Method of Fourth Order.

It uses predictor- corrector strategy which requires either four Runge-Kutta or Taylor's series sets of starting values in which the first set comes directly from the initial conditions while the other three sets are obtained from any of the two methods. The corresponding derivatives are.

$$y'_{n+1} = f(x_{n+1}, y_{n+1}, z_{n+1}) \quad 4$$

$$z'_{n+1} = g(x_{n+1}, y_{n+1}, z_{n+1}) \quad 5$$

Runge-Kutta method of order four is then applied as thus;

$$y_{n+1} = y_n + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

$$z_{n+1} = z_n + 1/6(l_1 + 2l_2 + 2l_3 + l_4)$$

and Adams-Moulton method became

Predictors: $py_{n+1} = y_n + h/24(55y'_n - 59y'_{n-1} + 37y'_{n-2} - 9y'_{n-3})$ 6

$pz_{n+1} = z_n + h/24(55z'_n - 59z'_{n-1} + 37z'_{n-2} - 9z'_{n-3})$ 7

Correctors: $y_{n+1} = y_n + h/24(9py'_{n+1} + 19y'_n - 5y'_{n-1} + y'_{n-2})$ 8

$z_{n+1} = z_n + h/24(9pz'_{n+1} + 19z'_n - 5z'_{n-1} + z'_{n-2})$ 9

Numerical Experiments

The level of stability of the two numerical methods were tested using two selected sample problems of second order DE with the application of C++ programming language under the same step size. Each of the numerical techniques result was presented in tabular form. The differential equations that had been used by some eminent scholars whose exact solutions are known on the interval $0<x<1.0$ and $1.0<x<2.0$ were used (Richard and Gabriel, 2006). The selected examples are computationally solved to illustrate the efficiency in terms of stability. The comparison of the solution of each of the methods was made with the exact solutions. The outputs were represented in tabular forms.

<u>Sample equation I</u>

Solve equation $y'' - 3y' + 2y = 0; y(0) = 1, y'(0) = 0, h = 0.1$

By defining $z = y'$, we have $z(0) = y'(0) = 0, and\ z' = y''$,

this differential equation can be rewritten as

$y'' = 3y' + -2y\ or\ z' = 3z - 2y$ we then obtain two first order system as

(a) $z = y'$

(b) $z' = 3z - 2y$, with $y(0) = 1, z(0) = 0$

Table 1 shown below presents the result of sample equation one in which column one is used for the number of iterations that were carried out and column two, three and four represent the approximated solutions of Runge-Kutta, Adams-Moulton and Exact methods respectively.

International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 4, December 2018
http://www.casirmediapublishing.com

Table 1. Approximated values for sample equation one

X	R-K values	AM values	Exact values
0.0	-1.00000000	-1.00000000	-1.0000000
0.1	-0.98894167	-0.98894167	-0.9889391
0.2	-0.95098718	-0.95098718	-0.9509808
0.3	-0.87761054	-0.87761054	-0.8775988
0.4	-0.80423389	-0.75812120	-0.7581085
0.5	-0.73085725	-0.57917393	-0.5781607
0.6	-0.65748061	-0.32413399	-0.3241207
0.7	-0.58410396	0.02768188	0.0276946
8	-0.51072732	0.50193957	0.5019506
0.9	-0.43735067	1.13043341	1.1304412
1.0	-0.36397403	1.95248984	1.9524924

Table 2 presents the error differences of sample equation one such that column one is for iterations and column two and three represent Runge-Kutta, Adams-Moulton error differences respectively. Figure one is the graph of the error differences to discover the level of stability graphically.

Table 2: Error differences between each method and exact solutions

Iteration (X)	R-K Error Differences	AM Error Differences
0.1	5.7E-07	5.7E-07
0.2	6.38E-06	6.38E-06
0.3	1.17E-05	1.17E-05
0.4	0.046125	1.27E-05
0.5	0.151697	1.32E-05
0.6	0.33336	1.33E-05
0.7	0.611799	1.27E-05
0.8	1.012678	1.1E-05
0.9	1.567792	7.79E-06
1.0	2.316466	2.56E-06

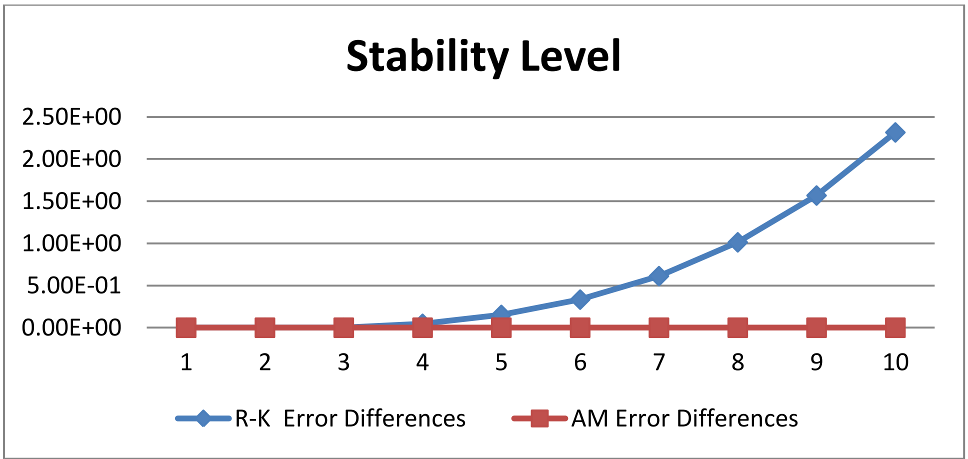

Figure 2: Graph of error differences between R-K and AM solutions

Sample equation II

Solve equation $3x^2 y'' - xy' + y = 0; y(1) = 4,\ y'(1) = 2, h = 0.1$

By defining $z = y'$, we have $z(1) = y'(1) = 2,\ y'(0) = 1, and\ z' = y''$,

this differential equation can be rewritten as

$y'' = (xy' - y)/3x^2\ or\ z' = (xz - y)/3x^2$ we then obtain two first order system as

(a) $z = y'$

(b) $z' = (xz - y)/3x^2$, with $y(1) = 4,\ z(1) = 2$

Table 3 presents the result of sample equation two in which column one is used for the number of iterations that were carried out and column two, three and four represent the approximated solutions of Runge-Kutta, Adams-Moulton and Exact methods respectively.

Table 3. Approximated values for sample equation two

X	R-K values	AM values	Exact values
1.0	4.00000000	4.00000000	4.0000000
1.1	4.19684019	4.19684019	4.1967235
1.2	4.38797544	4.38797544	4.3879757
1.3	4.57417829	4.57417829	4.5721050
1.4	4.76038114	4.75606990	4.7560668
1.5	4.94658399	4.93414769	4.9320142
1.6	5.13278683	5.10882722	5.1088213
1.7	5.31898968	5.28045588	5.2804560
1.8	5.50519253	5.44932745	5.4493212
1.9	5.69139538	5.58995912	5.5899750
2.0	5.87759823	5.77976854	5.7797632

Table 4 presents the error differences of sample equation two such that column one is for iterations and column two and three represent Runge-Kutta, Adams-Moulton error differences respectively. Figure two that is shown below is the graph of the error differences to discover the level of stability graphically.

Table 4: Error differences between each method and exact solutions

Iteration X	R-K Error Difference	AM Error Difference
1.1	-0.00611669	-0.00612
1.2	-0.00000026	-2.6E-07
1.3	-0.01207329	-0.01207
1.4	-0.00431434	-3.2E-05
1.5	-0.1456979	-0.00213
1.6	-0.02396553	-5.9E-06
1.7	-0.08193328	-0.0434
1.8	-0.05587133	-6.3E-06
1.9	-0.10142038	-0.02572
2.0	-0.09783503	-5.3E-06

International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 4, December 2018
http://www.casirmediapublishing.com

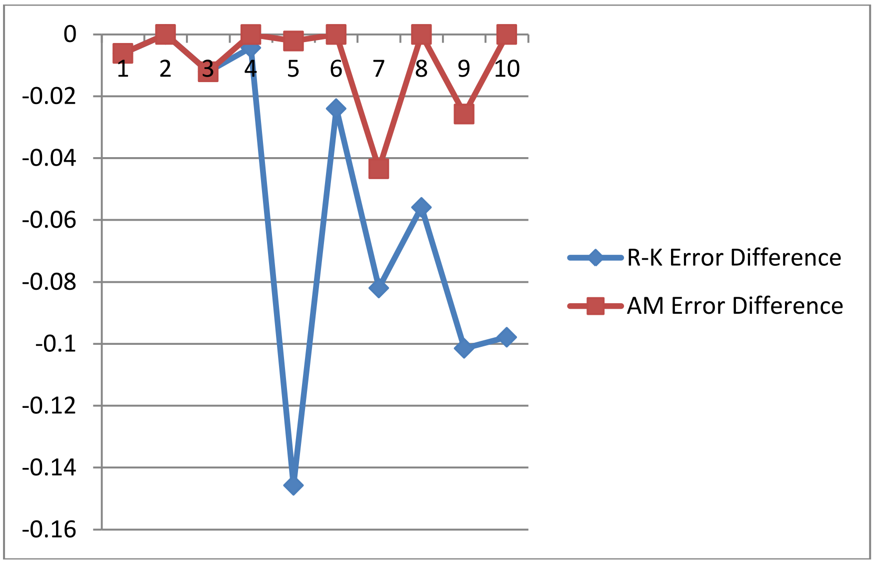

Figure 2: Graph of error differences between R-K and AM solutions

DISCUSSION OF THE FINDINGS

The results had shown that the approximated values of Adams-Moulton are so close to the exact solutions than Runge-Kutta values. Meanwhile, in Table 1, Adams values ranged between -1.000000 to 1.9524898 while Runges values ranged between -1.000000 to -0.3639740. In Table 3, their corresponding values ranged between 4.0000000 to 5.7797685 and 4.0000000 to 5.8775982 respectively. These Table 1 and 3 are therefore have been used to present the approximated values of the selected sample problems while Table 2 and 4 are presented to discover the error differences so far the paper is focused on comparing the level of stability of standard and multistep method using Runge-Kutta and Adams-Moulton techniques as the numerical algorithms. The error differences are used to determine the level of stability of these numerical techniques under considerations. Therefore, in the first sample problem, the error differences of Adam's method ranged between 5.7E-07 to 2.56E-06 while that of Runge's method ranged between 5.7E-07 to 2.3165. In the second sample problem, their corresponding error differences ranged between -6.12E-03 to -5.3E-06 and -6.12E-03 to -9.7835E-02 respectively. These error differences are graphically presented in Figure 1 and 2 because according to Cheney and Kincaid (1985), the differential equation gives rise to a family of solution curve, each corresponding to one value of parameter. And that stability can only be known through the curves derived from the numerical solutions based on the numerical techniques adopted. Therefore, the result presented in tabular and graphical forms had revealed that Adams-Moulton method is more stable if not absolutely stable than Runge-Kutta method. Meanwhile, by applying Grewal (2007) that any numerical method is said to be stable if it produces bounded solutions which imitate the exact solutions and by looking at the result realized in the two equations under consideration which are presented in Table 1 and 3, it is glaring that Adams values imitate the exact solutions and hence it is more stable than Runge-Kutta's values.

CONCLUSION

This work considered the level of stability of two numerical methods (Runge-Kutta and Adams) for solving ordinary differential equation of order two. It was observed that the two algorithms under consideration are good algorithms for solving ODE problems. However, the res ults obtained in the two samples proved that Adams-Moulton method is more stable than its counterpart when their approximated values and error differences were compared with exact solutions.

REFERENCES

Agam, O. and Odion, U (2012). Perturbed Runge-Kutta algorithms for solving linear and non-Linear Differential Equation. Journal of the Nigerian Association of Mathematical Physics, Volume 21.

Atajoromavwo, E.J. and Onavwie, U.A. (2012). Computer Approach to Runge-Kutta MethodOf First Order Differential Equation Using C++ Programming. Journal of the Nigerian Association of Mathematical Physics, Volume 21, pp 345-348

Butcher, J. C. (2003). Numerical algorithms for ordinary differential equation: Solution of First Order differential equation and High Order differential equation. http:// en.wikipedia.org/wiki/Numerical_methods_for_ordinary_differ.... 9/29/2012

Cheney, W. and Kincaid, D. (1985). Numerical Mathematics and Computing, 2nd Edition Brooks/cole Publishing Company. ISBN 0-534-04356-9. P 305

Gilberto E.U (2010). The 4th – Order Runge-Kutta Method for a 2nd order ODE. January, 2010

Grewal, B.S. (2007). Numerical Methods in Engineering and science with program in Fortran 77, C, C++. Published by Romesh Chander Khana. ISBN: 81-7409-205-6. Pp 225-257.

John, H. M. (1995). Numerical Methods for Mathematics, Science and Engineering. Second edition, New Jersey, USA International Edition. P 471

Kendall, E.A. (2007).Theoretical Numerical Analysis: a Functional Analysis Framework, 2nd Edition. Scholarpedia, 2(8): 3163

Richard B. and Gabriel B. (2006): Schaum's outline of Differential Equations, 3rd edition. Schaum's outline series, McGraw-Hill. Pp 195-205.

Siddiqi, A.H and Manchanda, P. (2006): A first course in Differential equations with Applications. Published by Rajiv Beri for Macmillan India Ltd.

Steven, C.C. and Raymond, P.C. (2010). Numerical Methods for Engineers, 6th edition. Published by McGraw Hill Higher Education. www.mhhe.com ISBN: 978-007-126759-5 pp733-773

http://farside.ph.utexas.edu/teaching/329/lectures/node35.html 9/29/2012.

http://math.fullerton.edu/mathews/n2003/adamsbashforthmod.html 1/7/2013

http://math.fullerton.edu/mathews/n2003/RungeKuttamod.html 1/7/2013

http://en.wikipedia.org/wiki/Numerical_methods5/8/2014

Numerical analysis-wikipedia, the free encyclopedia 5/8/2014

International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 4, December 2018
http://www.casirmediapublishing.com

Numerical method for ordinary differential equations – Wikipedia, the---- http://en.wikipedia.org/wiki/Numeical_methods_for_ordinary_differ…. 1/11/2016
http:/www.scholarpedia/org/article/numerical_analysis pp 1-8, 1/11/2016