International Journal of Science and Advanced Innovative Research
ISSN: 2536-7315 (Print) 2536-7323 (Online)
Volume 3, Number 2, June 2018
http://www.casirmediapublishing.com

LAPLACE TRANSFORM SOLUTION OF HIGHER ORDER INITIAL VALUE PROBLEMS

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ABSTRACT

This paper considers the derivation of higher order Laplace transforms. The derivations were obtained from lesser orders and progressively to the sixth order and then generalized. The higher order derivations were then used in the solution of higher order initial value problems (IVP) of orders three to six to illustrate the robustness of Laplace transform solution of ODEs. The solutions were consistent with the given equations which show that the method is valid for higher order IVPs.
***Key Words**: Laplace Transform, Ordinary Differential Equations, Initial Value Problems*

INTRODUCTION

Differential equations arise from the modeling of physical problems in several fields. The solution technique adopted varies as the problem. The analytic and numerical solution Methods of ODEs have been extensively dealt with in several literatures.

Consider the solution of the nth order initial-value problem

$$y^{(n)} = f\left(x, y, y', y'', \cdots, y^{(n-1)}\right) \quad (1.1)$$

with the initial conditions

$$y(0) = y_0,\ y'(0) = y'_0,\ \cdots\ y^{(n-1)}(0) = y_0^{(n-1)} \text{ [3]} \quad (1.2)$$

An nth order linear ordinary differential equation (ODE) of the form

$$af^{(n)}(x) + bf^{(n-1)}(x) + cf^{(n-2)}(x) + ... + qf(x) = w(x) \quad (1.3)$$

Where $a, b, c, ..., q$ are constants or functions of x can be solved if a general form of f is obtained with its associated integration constants which can be obtained when
The given boundary conditions are applied. [6], [1].

Laplace transform provides an efficient way of solving (1) by applying the boundary conditions from the onset. The method converts the given ODE into an algebraic equation. The algebraic equation so derived is then solved algebraically and converting the solution via the table of inverse

Laplace transform. Again Laplace transform is more efficient in solving problems represented by discontinuous functions. [6].

In this paper, we consider the solution of higher order initial-value problems (IVPs) by Laplace transform.

Definition: The Laplace transform of a function $f(t)$ is the half range non-negative integral defined as

$$L\{f(t)\}=\int_0^{\infty} e^{-tx} f(t)\,dt = F(s) \quad (1.4)$$

Where $f(t)$ is a piecewise continuous function.

Sufficient Condition: A sufficient condition for the existence of Laplace transform is that

(a) $f(t)$ is piecewise continuous

(b) $f(t)$ is an exponential of order $'a'$

Lemma: A function $f(t)$ has a Laplace transform if for all $t \geq 0$, $|f(t)| \leq \alpha e^{kt}$ for some constants α and k.

Theorem: If $f(t)$ is defined and piecewise continuous on every finite interval on the semi axis $t \geq 0$ and some constants α and k, then the Laplace transform $L\{f\}$ exists for all $s > k$ [5].

MATERIALS AND METHODS

The Laplace transform of the function $f(t)$ is defined as (2) above

Hence the Laplace transform of the function $f'(t)$ be defined as

$$L\{f'(t)\}=\int_0^{\infty} e^{-st} f'(t)\,dx \quad (2.1)$$

Integrating (3) by parts yields

$$L\{f'(t)\}=e^{-st} f(t)\Big|_0^{\infty} + s\int_0^{\infty} f(t)dt \quad (2.2)$$

$$= e^{-st} f(t)\Big|_0^{\infty} + sF(s)$$

$$= sF(s) - f(0) \qquad (2.3)$$

Let $g(t) = f'(t)$ then $g'(t) = f''(t)$ and $g'(0) = f''(0)$

We then define as in (4)

$$L\{f''(t)\} = \int_0^\infty e^{-st} f''(t)\,dt$$

Let $L\{g(t)\} = L\{f'(t)\} = G(s)$ then we have

$$L\{g'(t)\} = L\{f''(t)\} = sG(s) - g(0)$$

$$L\{f''(t)\} = s\left[sF(s) - f(0)\right] - f'(0)$$

$$= s^2F(s) - sf(0) - f'(0) \quad (2.4)$$

Next, we define

$$L\{h(t)\} = L\{g'(t)\} = H(s)$$

$$L\{h'(t)\} = L\{g''(t)\} = sH(s) - h(0)$$

$$= sL\{g'(x)\} = s\left[s^2F(s) - sf(0) - f'(0)\right] - f''(0)$$

$$= s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$L\{f'''(x)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0) \quad (2.5)$$

Again we define

$$L\{m(t)\} = L\{h'(t)\} = M(s)$$

$$L\{m'(t)\} = L\{h''(t)\} = sM(s) - m(0)$$

$$= sL\{m'(x)\} = s\left[s^3F(s) - s^2f(0) - sf'(0) - f''(0)\right] - f'''(0)$$

$$L\left\{f^{(iv)}(x)\right\} = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0) \quad (2.6)$$

which is the Laplace transform of a fourth derivative and generalize for the nth order as

$$L\left\{f^{(n)}(x)\right\} = s^nF(s) - s^{(n-1)}f(0) - s^{(n-2)}f'(0) - s^{(n-3)}f''(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

(2.7)

Hence if $f^{(n)}$ is piecewise continuous and the functions f, f', $f'' \cdots$, $f^{(n-1)}$ are all continuous, then (6) holds [2].

The solution technique implies that, there must exist also an inverse

RESULTS AND DISCUSSIONS

In this section, we consider four examples of linear ordinary differential equations of orders three, four, five and six in that order, all of which are initial-value problems (IVPs).

Example 1:

$$y''' + y' = -2e^{-x}, \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = 1$$

Solution:

Given $y'''+y'=-2e^{-x}$ and the initial condition $y(0)=1, y'(0)=0, \ y''(0)=1$ we take the Laplace transform to have

$L\{y'''\}+L\{y'\}=-2L\{e^{-x}\}$

This gives

$$\left[s^3F(s)-s^2y(0)-sy'(0)-y''(0)\right]+\left[sF(s)-y(0)\right]=\frac{-2}{s+1}$$

Substituting the initial conditions yields

$$\left[s^3F(s)-s^2-1\right]+\left[sF(s)-1\right]=\frac{-2}{s+1}$$

$$s^3F(s)+sF(s)-s^2-2=\frac{-2}{s+1}$$

$$\left(s^3-s\right)F(s)=s^2+2-\frac{2}{s+1}$$

$$F(s)=\frac{s}{\left(s^2-1\right)}+\frac{2}{s\left(s^2-1\right)}-\frac{2}{s(s+1)\left(s^2-1\right)}$$

$$=\frac{s^3+s^2+2s}{s+1}$$

Resolving into partial fractions gives

$$F(s)=\frac{1}{s+1}+\frac{1}{s^2+1}$$

Taking the inverse Laplace transform gives

$$f(x)=L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=e^{-x}+\sin x$$

Example 2:

$y^{(iv)}=4y''-3e^{-x}, \ y(0)=2, \ y'(0)=1, \ y''(0)=5, \ y'''(0)=7$

Solution

$$L\left\{y^{(iv)}\right\}=4L\{y''\}-3L\{e^{-x}\}$$

$$\left[s^4F(s)-s^3y(0)-s^2y'(0)-sy''(0)-y'''(0)\right]=4\left[s^2F(s)-sy(0)-y'(0)\right]-\frac{3}{s+1}$$

$$\left[s^4F(s)-2s^3-s^2-5s-7\right]=4s^2F(s)-2s-1-\frac{3}{s+1}$$

$$s^4F(s)-2s^3-s^2-5s-7-4s^2F(s)+2s+1+\frac{3}{s+1}$$

$$\left(s^4-4s^2\right)F(s)=2s^3+s^2-3s+7-3+\frac{3}{s+1}$$

$$\left(s^4-4s^2\right)F(s)=\frac{2s^4+3s^3-2s^3}{s+1}$$

$$F(s)=\frac{2s^4+s^3}{(s+1)\left(s^4-4s^2\right)}$$

$$F(s)=\frac{s(2s+1)}{(s+1)(s-2)(s+2)}$$

Resolving into partial fractions gives

$$F(s)=\frac{1}{s+1}+\frac{1}{s-2}$$

Taking the inverse Laplace transform gives

$$f(x)=L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{1}{s-2}\right\}$$

$$=e^{-x}+e^{2x}$$

Example 3

$$y^{(v)}-y'=30e^{2x},\ \ y(0)=3,\ y'(0)=0,\ \ y''(0)=y'''(0)=6,\ \ y^{(iv)}(0)=18$$

$$L\left\{y^{(v)}\right\}-L\left\{y'\right\}=30L\left\{e^{2x}\right\}$$

$$\left[s^5F(s)-s^4y(0)-s^3y'(0)-s^2y''(0)-sy'''(0)-y^{(iv)}(0)\right]-\left[sF(s)-y(0)\right]=\frac{30}{s-2}$$

$$\left[s^5F(s)-3s^4-6s^2-6s-18\right]-\left[sF(s)-3\right]=\frac{30}{s-2}$$

$$s^5F(s)-3s^4-6s^2-6s-18-sF(s)+3=\frac{30}{s-2}$$

$$\left(s^5-s\right)F(s)=3s^4+6s^2+6s+15+\frac{30}{s-2}$$

$$\left(s^5-s\right)F(s)=3s^4+6s^2+6s+15+\frac{30}{s-2}$$

Resolving into partial fractions gives

$$F(s)=\frac{1}{s-2}+\frac{2}{s+1}$$

Taking the inverse Laplace transform

$$f(x)=L^{-1}\left\{F(s)\right\}=L^{-1}\left\{\frac{1}{s-2}\right\}+L^{-1}\left\{\frac{2}{s+1}\right\}$$

$$f(x)=e^{2x}+2e^{-x}$$

Example 4

$$y^{(vi)} + 2y^{(v)} = 64e^{2x},$$

$$y(0) = y'(0) = 1, \quad y''(0) = 4, \quad y'''(0) = 0, \quad y^{(iv)}(0) = 16, \quad y^{(v)}(0) = 0$$

$$L\{y^{(vi)}\} + 2L\{y^{(v)}\} = 64L\{e^{2x}\}$$

$$\left[s^6F(s) - s^5y(0) - s^4y'(0) - s^3y''(0) - s^2y'''(0) - sy^{(iv)}(0) - y^{(v)}\right]$$

$$+2\left[s^5F(s) - s^4y(0) - s^3y'(0) - s^2y''(0) - sy'''(0) - y^{(iv)}(0)\right] = \frac{64}{s-2}$$

$$s^6F(s) - s^5 - s^4 - 4s^3 - 16s + 2\left[s^5F(s) - s^4 - s^3 - 4s^2 - 16\right] = \frac{64}{s-2}$$

$$s^5(s+2)F(s) = s^5 + 3s^4 + 6s^3 + +8s^2 + 16s + 32 + \frac{64}{s-2}$$

$$F(s) = \frac{s^6 + s^5 - 4s^3}{s^5(s+2)(s-2)}$$

$$F(s) = \frac{s^3 + s^2 - 4}{s^2(s+2)(s-2)}$$

Resolving into partial fractions gives

$$F(s) = \frac{1}{s^2} + \frac{1}{2}\left(\frac{1}{s-2}\right) + \frac{1}{2}\left(\frac{1}{s+2}\right)$$

Taking the inverse Laplace transform

$$f(x) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{1}{2}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$f(x) = x + \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$$

$$f(x) = x + \cosh e^{2x}$$

CONCLUSION

The solution of initial value problems using Laplace transform has been demonstrated for higher order differential equations (ODEs). The simplicity of the method is seen in the incorporation of the initial conditions at the onset which reduces the ODE to an algebraic equation, thus making it time efficient as the results obtained can be read from inverse Laplace transform tables. Also Laplace transform gives the exact solutions of the given problems.

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